PSI Center for Photon Science

EPFL

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Chapter 2 - Fourier optics and imaging

PHYS-715 Physical Optics and Advanced Imaging

CXI

Course contents

Cover the unifying concepts of imaging, including with electrons, optics, X-rays. In 2D, 3D, and beyond.

Chapter 1 Interactions of waves with the sample and beyond

- Review of Fourier transform and properties
- Helmholtz equation
- Angular spectrum and evanescent waves
- Huygens–Fresnel principle
- Rayleigh-Sommerfeld solution
- Fresnel approximation
- Fraunhofer approximation – the far field
- Paraxial wave equation
- Projection approximation
- Multislice propagation
- Beer-Lambert law

Recap on propagation

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Angular spectrum

$$U(x,y,z) = \mathcal{F}^{-1}\left\{\mathcal{F}\left\{U(x,y,0)\right\} H(f_x, f_y)\right\}$$

$$H(f_x, f_y) = \exp\left[i2\pi z\sqrt{\left(\frac{k_0}{2\pi}\right)^2 - f_x^2 - f_y^2}\right]$$

Rayleigh-Sommerfeld

$$U(P_0) = U(P_1) * h(x_1, y_1)$$

$$h(x,y) = \frac{1}{i\lambda_0}\left(1 - \frac{1}{ik_0 r}\right)\frac{\exp(ikr)}{r}\frac{z}{r}$$

J. W. Goodman, Introduction to Fourier Optics, fourth edition. McMillan learning (2017)

Fresnel approximation

Convolution form

$$U_o(x,y) = U_i(x,y) * h(x,y)$$

$$h(x,y) = \frac{\exp(ik_0 z)}{i\lambda_0 z}\exp\left(\frac{ik_0}{2z}\left[x^2+y^2\right]\right)$$

$$\tilde{U}_0(f_x,f_y) = \tilde{U}_i(f_x,f_y)H(f_x,f_y)$$

$$H(f_x,f_y) = \exp(ik_0 z)\exp\left[-i\pi\lambda_0 z\left(f_x^{\,2}+f_y^{\,2}\right)\right]$$

Single Fourier transform form

$$U_o(x,y) = \frac{\exp(ik_0 z)}{i\lambda_0 z}\exp\left[\frac{ik_0}{2z}\left(x^2+y^2\right)\right]\mathcal{F}\left\{U_i(\xi,\eta)\exp\left[\frac{ik_0}{2z}\left(\xi^2+\eta^2\right)\right]\right\}_{\substack{f_x=\frac{x}{\lambda_0 z}\\ f_y=\frac{y}{\lambda_0 z}}}$$

J. W. Goodman, Introduction to Fourier Optics, fourth edition. McMillan learning (2017)

Projection approximation

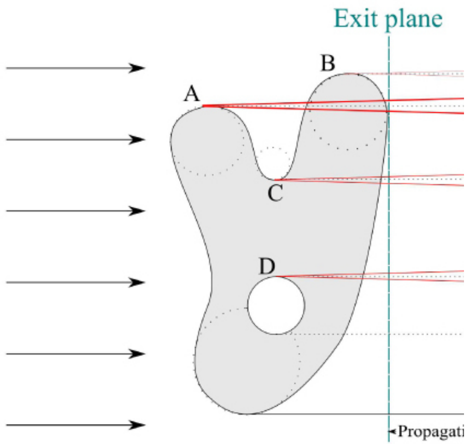
The object is thin enough, and boring enough, that we can ignore any diffraction within it

$$\tilde{\psi}(x,y,z=z_1) \approx \tilde{\psi}(x,y,z=z_0)\,o(x,y)$$

$$o(x,y) = \exp\left(ik_0 z\right)\exp\left\{ik_0 \int_0^z [n(x,y,z)-1]dz\right\}$$

$$n = 1-\delta+i\beta$$

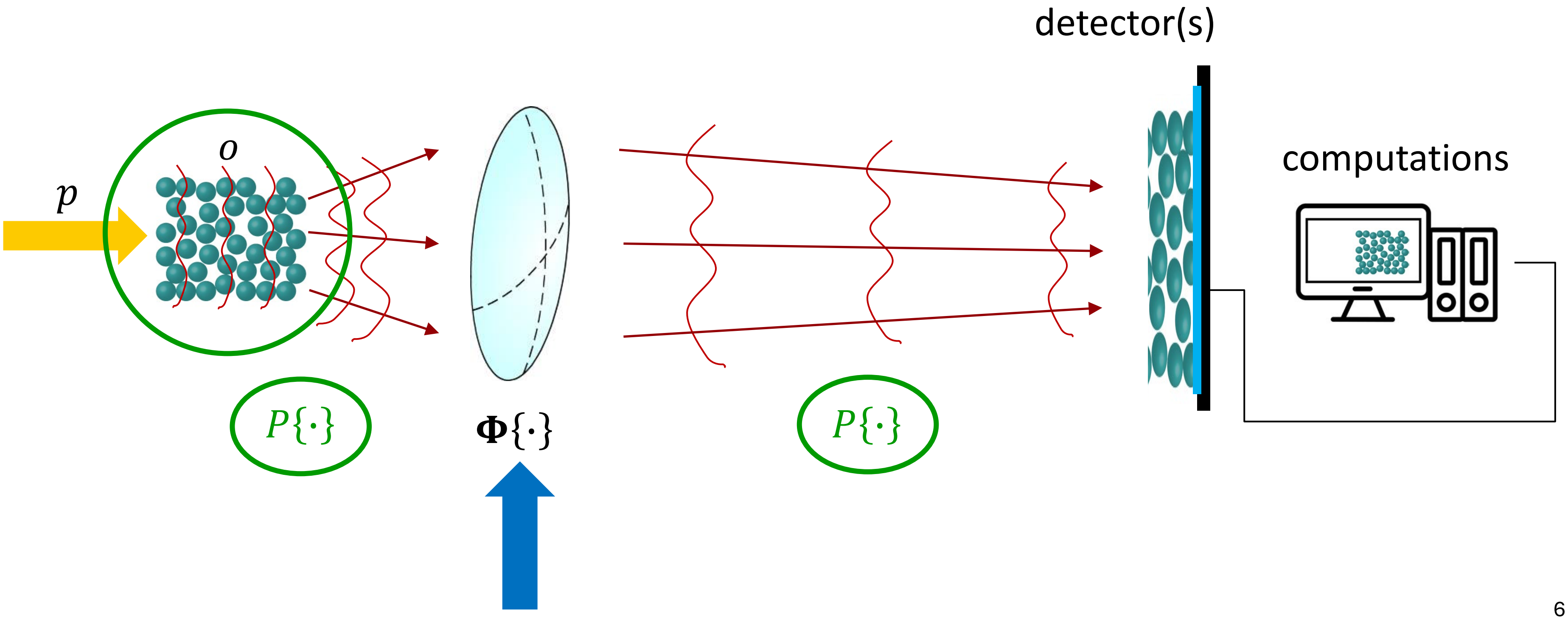
Index of refraction is complex-valued



K. S. Morgan, K. K. W. Siu, and D. M. Paganin, "The projection approximation and edge contrast for x-ray propagation-based phase contrast imaging of a cylindrical edge," Opt. Express **18**, 9865 (2010).

Lenses and imaging

A more detailed model



Course contents

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Cover the unifying concepts of imaging, including with electrons, optics, X-rays.
In 2D, 3D, and beyond.

Chapter 2 - Fourier Optics and Imaging

- Lens transmissivity
- Fourier transforming properties of lenses
- Focused beams
- Imaging
- Frequency response of optical systems (coherent and incoherent cases)
- Wavefront aberrations
- Propagation through complicated paraxial optical systems

Lens transmissivity

Types of lenses

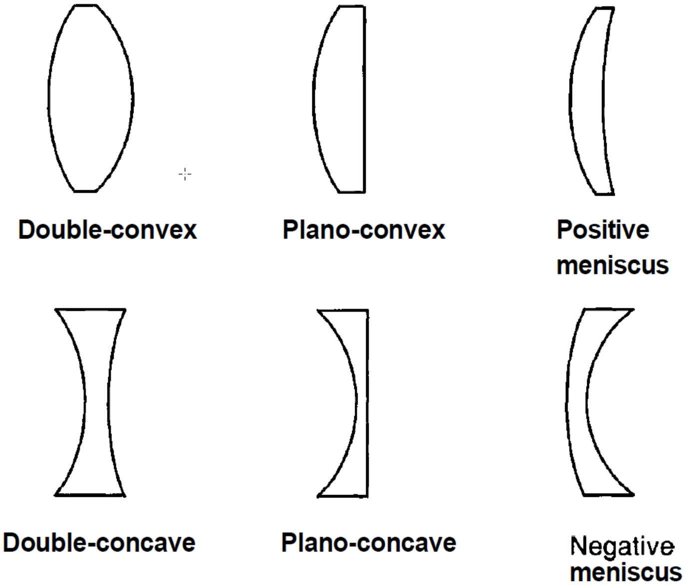
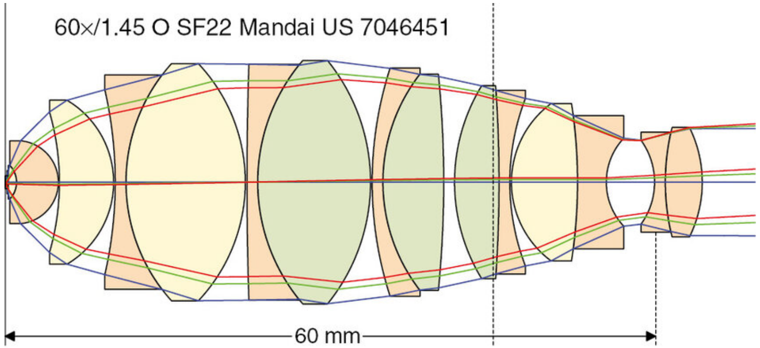


FIGURE 5.3
Various types of lenses.



J. W. Goodman, Introduction to Fourier Optics, fourth edition. McMillan learning (2017)

Types of lenses

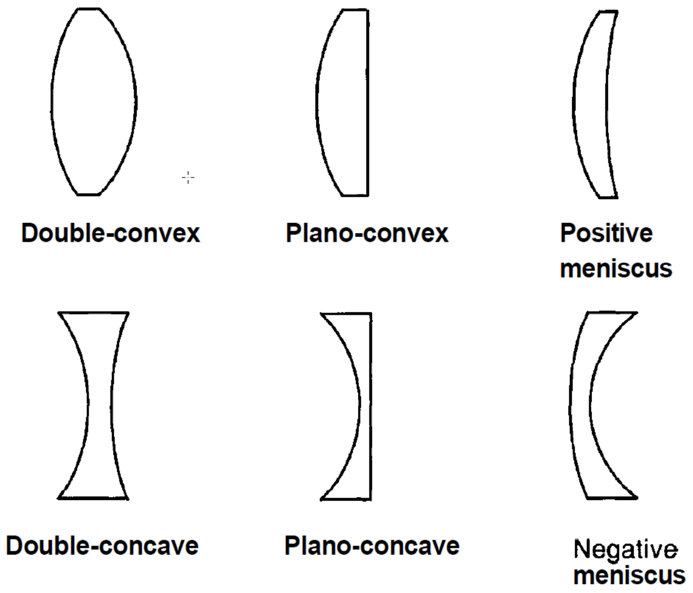
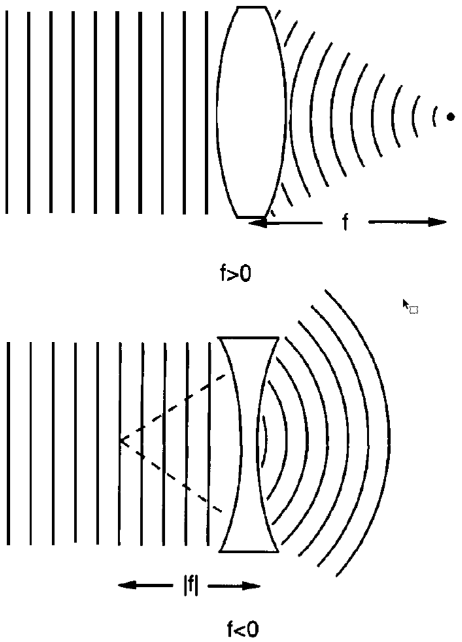


FIGURE 5.3
Various types of lenses.

$\mathrm{Re}\{n\} > 1$



J. W. Goodman, Introduction to Fourier Optics, fourth edition. McMillan learning (2017)

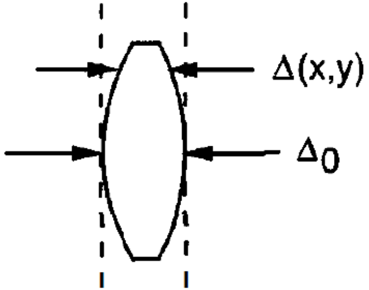
The thin lens transmissivity

Projection approximation (thin lens)

$$t_l(x,y) = \cancel{\exp\left(ik_0\Delta_0\right)} \exp\left\{ik_0\int_{z_0}^{z_1}[n(x,y,z)-1]dz\right\}$$

$n = 1-\delta+i\beta$

Index of refraction is complex-valued



$$t_l(x,y) = \exp\{ik_0\mathrm{OPD}(x,y)\} = \exp\{i\phi(x,y)\}$$

$$\phi = k_0\int_{z_0}^{z_1}[n(x,y,z)-1]dz$$

$$\phi = k_0\int_{z_0}^{z_1}[n(x,y,z)-1]dz = k_0(n-1)\Delta(x,y)$$

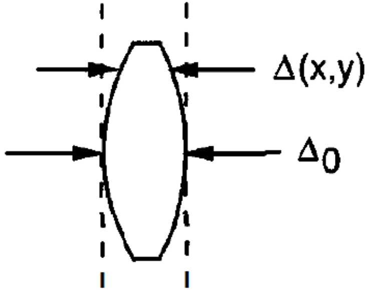
Note it is different from Goodman's by the constant phase factor.

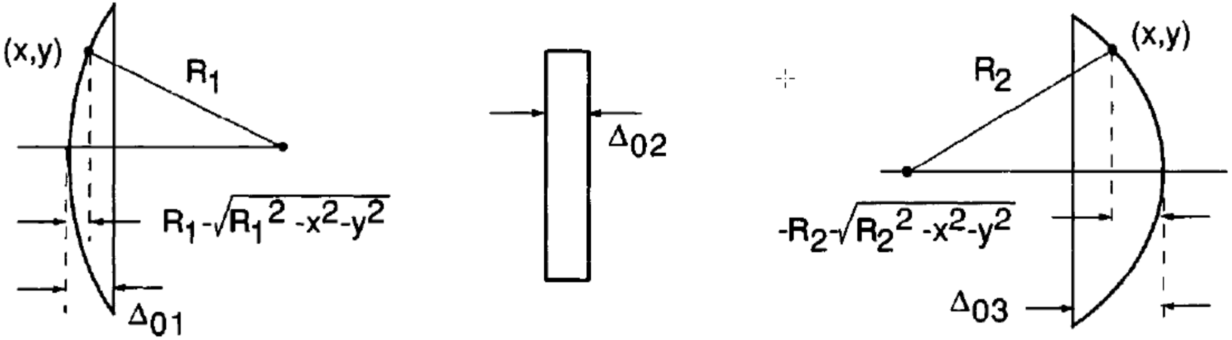
J. W. Goodman, Introduction to Fourier Optics, fourth edition. McMillan learning (2017)

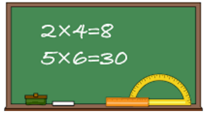
The thin lens transmissivity

$$U_l'(x,y) = t_l(x,y)U_l(x,y)$$

$$t_l(x,y) = \exp\{ik_0(n-1)\Delta(x,y)\}$$







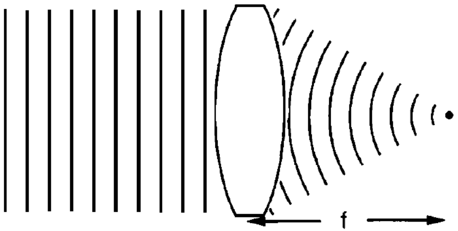
J. W. Goodman, Introduction to Fourier Optics, fourth edition. McMillan learning (2017)

The thin lens transmissivity

Paraxial approximation

$$\Delta(x,y) = \Delta_0 - \frac{x^2+y^2}{2}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

$$t_l(x,y) = \exp\{ik_0(n-1)\Delta(x,y)\}$$

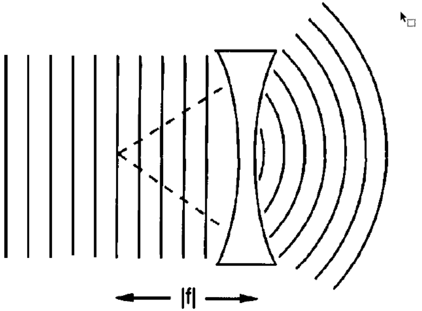


f>0

Lens-maker equation for a thin lens

$$\frac{1}{f} \approx (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

$$t_l(x,y) = \exp\left\{-\frac{ik_0}{2f}(x^2+y^2)\right\}$$



f<0

J. W. Goodman, Introduction to Fourier Optics, fourth edition. McMillan learning (2017)

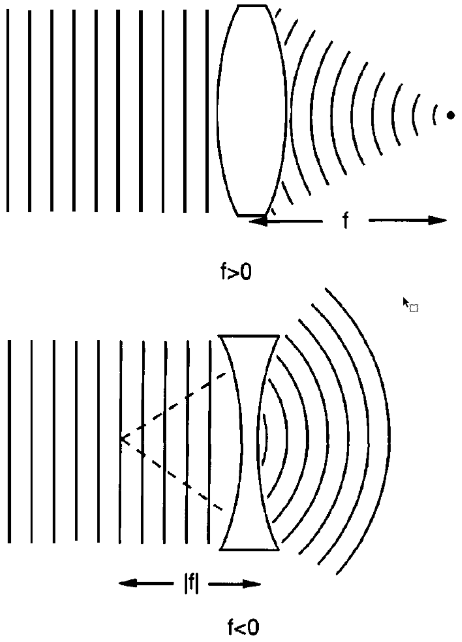
The thin lens transmissivity

For input plane wave

$$U_l'(x,y) = \exp\left\{-\frac{ik_0}{2f}(x^2+y^2)\right\}\cdot 1$$

Point source in Fresnel approximation

$$h(x,y) = \frac{\exp(ik_0 z)}{i\lambda_0 z}\exp\left(\frac{ik_0}{2z}\left[x^2+y^2\right]\right)$$



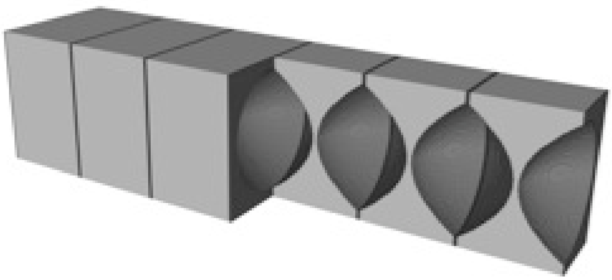
J. W. Goodman, Introduction to Fourier Optics, fourth edition. McMillan learning (2017)

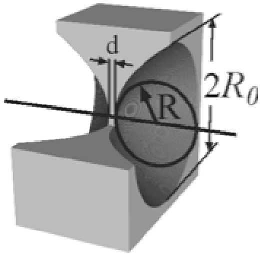
Side note for X-rays

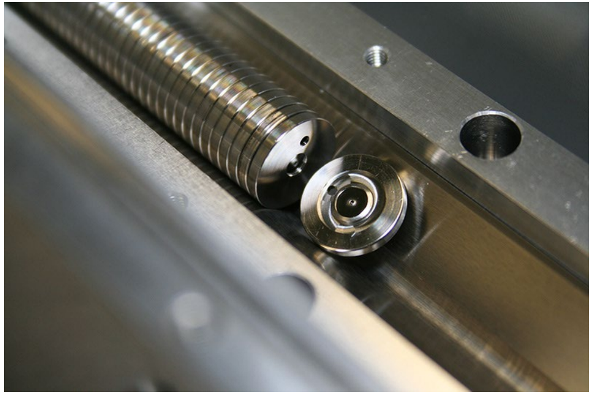
Phase velocity of the wave is larger than *c*

X-ray compound refractive lenses

$$\mathrm{Re}\{n\} < 1$$





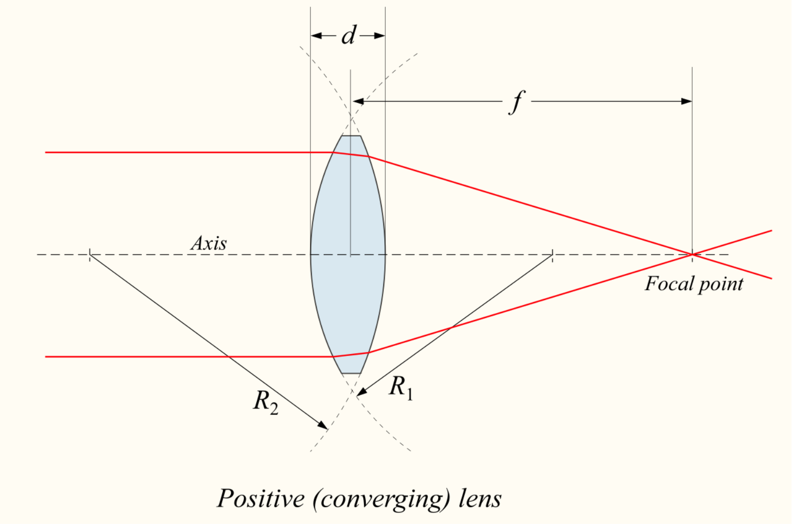


Stack of beryllium compound refractive lenses

https://photon-science.desy.de/research/research_teams/fs_petra/overview/x_ray_optics/refractive_x_ray_lenses/index_eng.html
https://photon-science.desy.de/research/research_teams/fs_petra/overview/x_ray_optics/rotationally_parabolic_refractive_lenses/index_eng.html

Side note for thick lenses

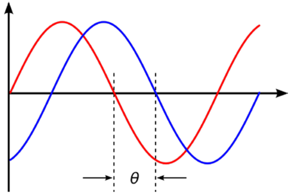
$$\frac{1}{f} = (n-1)\left[\frac{1}{R_1} - \frac{1}{R_2} + \frac{(n-1)d}{nR_1R_2}\right]$$



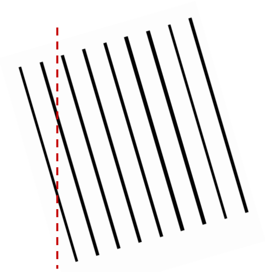
E. Hecht, Optics, 5th edition. Pearson (2017)
Image https://en.wikipedia.org/wiki/Lens#/media/File:Lens1.svg

Phase of a wavefield

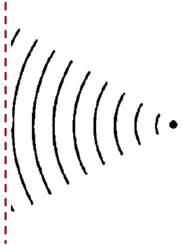
Phase differences describe relative offset of waves



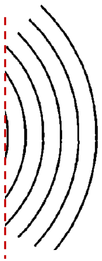
Represent wavefields by lines of constant phase



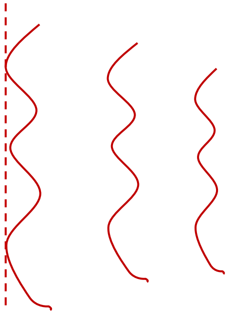
Plane-wave



Converging



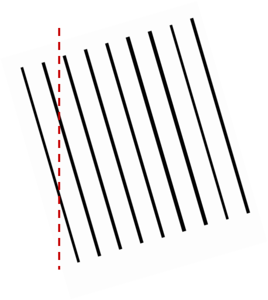
Diverging



Miscelaneous

https://en.wikipedia.org/wiki/Phase_(waves)

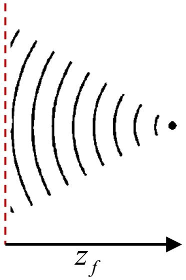
Phase of a wavefield



Plane-wave

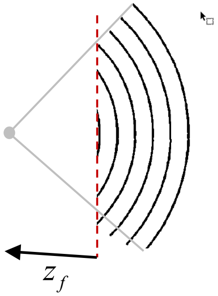
$$U(x,y) = A\exp\left[i(k_x x + k_y y)\right]$$

Linear phase



Converging

$$U_o(x,y) = \exp\left[-\frac{ik_0}{2z_f}\left(x^2 + y^2\right)\right]$$



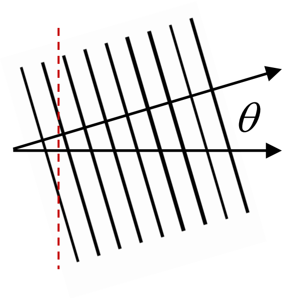
Diverging

$$U_o(x,y) = \exp\left[\frac{ik_0}{2z_f}\left(x^2 + y^2\right)\right]$$

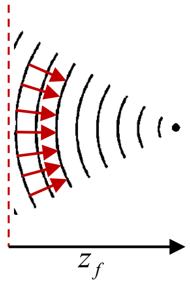
Quadratic phase

https://en.wikipedia.org/wiki/Phase_(waves)

Local derivative of the phase



Plane-wave



Converging

$$U(x,y) = A\exp\left[i(k_x x + k_y y)\right]$$

$$\frac{\partial\phi}{\partial y} = k_x = \frac{2\pi}{\lambda}\sin\theta$$

$$\sin\theta = \frac{\lambda}{2\pi}\frac{\partial\phi}{\partial y}$$

$$U_o(x,y) = \exp\left[-\frac{ik_0}{2z_f}\left(x^2+y^2\right)\right]$$

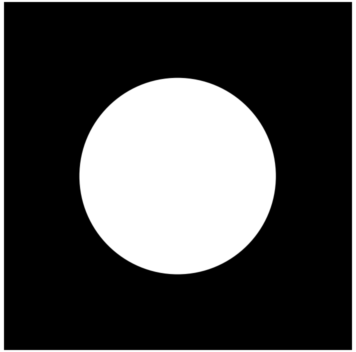
$$\sin\theta = \frac{\lambda}{2\pi}\frac{\partial}{\partial y}\left[-\frac{k_0}{2z_f}\left(x^2+y^2\right)\right] = -\frac{x}{z_f}$$

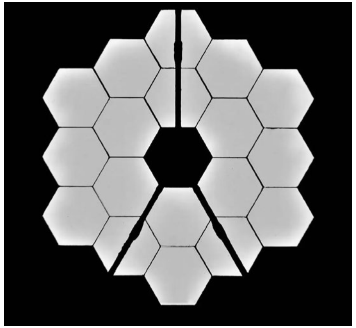
The aperture

Account for the limited physical extent of the lens via a pupil function

$$t_l(x,y) = P(x,y)\exp\left\{-\frac{ik_0}{2f}(x^2+y^2)\right\}$$

$$P(x,y) = \begin{cases} 1 & \text{inside the lens aperture} \\ 0 & \text{otherwise} \end{cases}$$





J. W. Goodman, Introduction to Fourier Optics, fourth edition. McMillan learning (2017)
Image from https://www.nasa.gov/image-article/mirrors-aligned-webb-telescopes-first-full-color-images-due-july/

Fourier transforming properties of lenses

Fourier transform using a lens

One of the most remarkable and useful properties of a converging lens is its inherent ability to perform two-dimensional Fourier transforms. This complicated analog operation can be performed with extreme simplicity in a coherent optical system, taking advantage of the basic laws of propagation and diffraction of light.

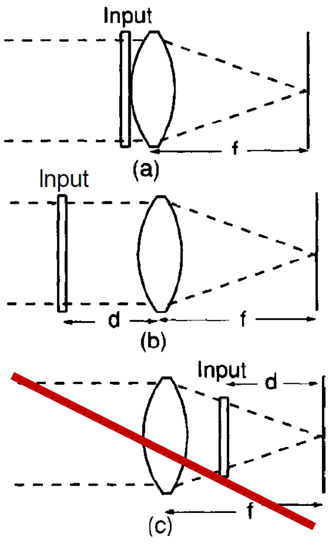
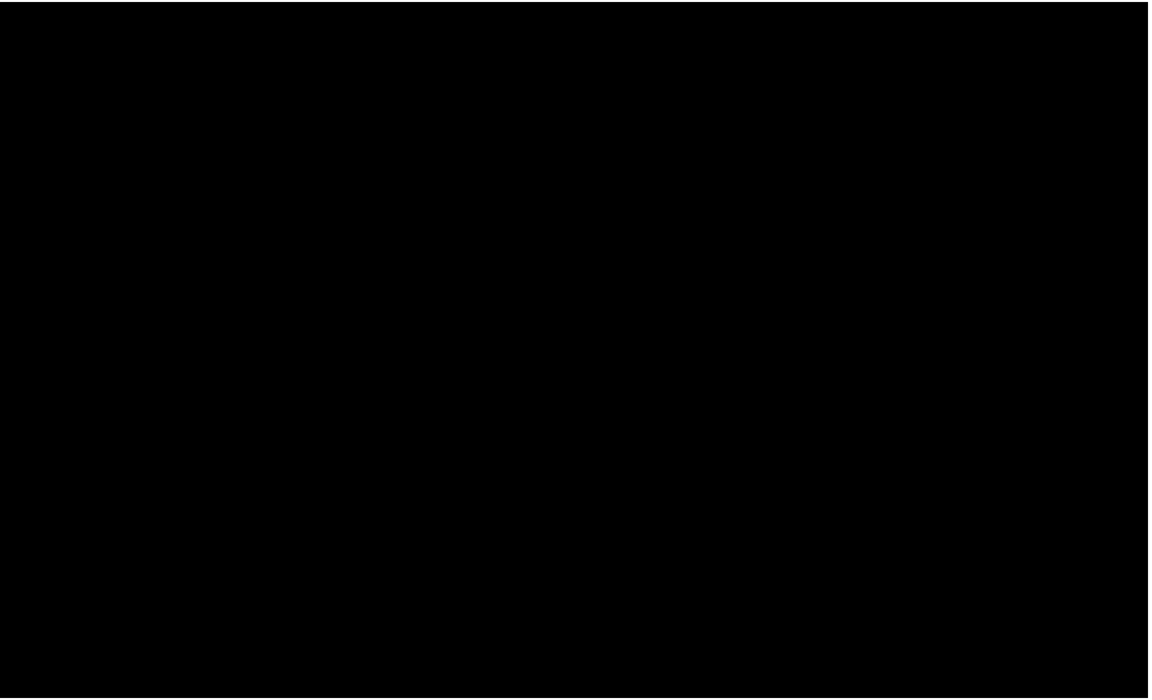


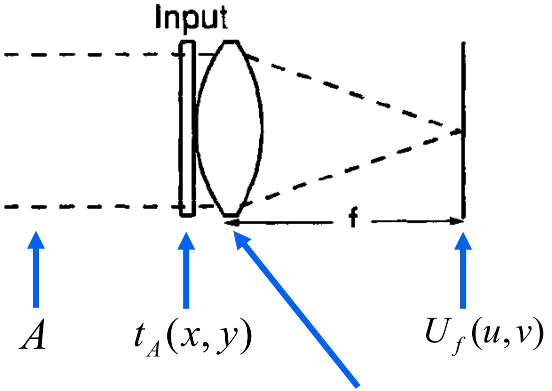
FIGURE 5.5
Geometries for performing the Fourier transform operation with a positive lens.

J. W. Goodman, Introduction to Fourier Optics, fourth edition. McMillan learning (2017)

Fourier transform using a lens case (a)

Object directly against the lens





$$t_l(x,y) = P(x,y)\exp\left\{-\frac{ik_0}{2f}(x^2+y^2)\right\}$$

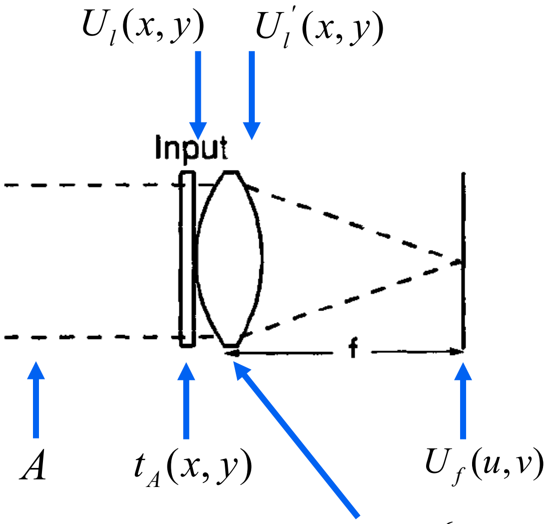
J. W. Goodman, Introduction to Fourier Optics, fourth edition. McMillan learning (2017)

Fourier transform using a lens case (a)

Object directly against the lens

$$U_f(u,v) = \frac{A}{i\lambda f} \exp\left\{ \frac{ik_0}{2f}(u^2+v^2) \right\} \mathcal{F}\left\{ t_A(x,y) P(x,y) \right\}_{\substack{f_x = \frac{u}{\lambda f} \\ f_y = \frac{v}{\lambda f}}}$$

$$U_f(u,v) = \frac{1}{i\lambda f} \exp\left\{ \frac{ik_0}{2f}(u^2+v^2) \right\} \mathcal{F}\left\{ U_l(x,y) P(x,y) \right\}_{\substack{f_x = \frac{u}{\lambda f} \\ f_y = \frac{v}{\lambda f}}}$$



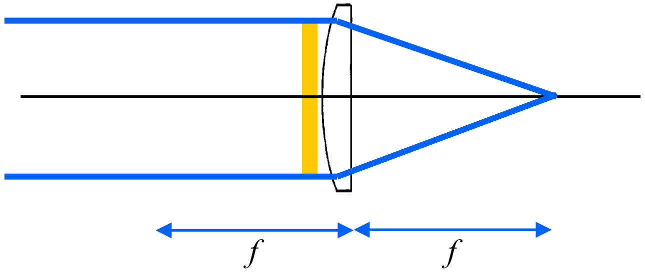
$$t_l(x,y) = P(x,y) \exp\left\{ -\frac{ik_0}{2f}(x^2+y^2) \right\}$$

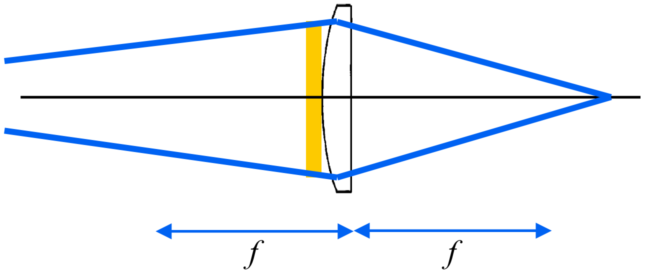
J. W. Goodman, Introduction to Fourier Optics, fourth edition. McMillan learning (2017)

Focused beams

Focused beams

No object corresponds to focusing a beam → Beam at focus is the FT of the pupil





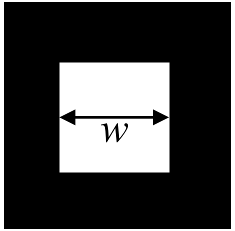
The focused beam – square aperture

Beam at the focus will be equal to the FT of the pupil $t_A(x,y)=1$

$$U_f(u,v)=\frac{A}{i\lambda f}\exp\left\{\frac{ik_0}{2f}(u^2+v^2)\right\}\mathcal{F}\left\{t_A(x,y)P(x,y)\right\}_{\substack{f_x=\frac{u}{\lambda f}\\ f_y=\frac{v}{\lambda f}}}$$

For square pupil

$$P(x,y)=\mathrm{rect}\left(\frac{x}{w}\right)\mathrm{rect}\left(\frac{y}{w}\right)$$



$$\mathcal{F}\left\{P(x,y)\right\}=w^2\,\mathrm{sinc}\left(wf_x\right)\mathrm{sinc}\left(wf_y\right)$$

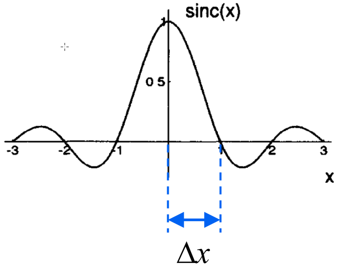
$$U_f(u,v)=\frac{1}{i\lambda f}\exp\left\{\frac{ik_0}{2f}(u^2+v^2)\right\}w^2\,\mathrm{sinc}\left(\frac{wu}{\lambda f}\right)\mathrm{sinc}\left(\frac{wv}{\lambda f}\right)$$

The focused beam – square aperture

For square pupil $$P(x,y) = \mathrm{rect}\left(\frac{x}{w}\right)\mathrm{rect}\left(\frac{y}{w}\right)$$

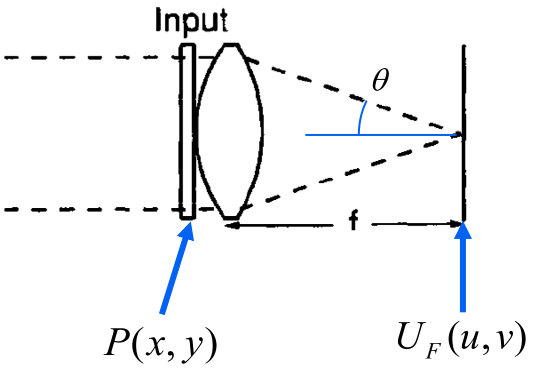
$$U_f(u,v) = \frac{1}{i\lambda f}\exp\left\{\frac{ik_0}{2f}(u^2+v^2)\right\} w^2\,\mathrm{sinc}\left(\frac{wu}{\lambda f}\right)\mathrm{sinc}\left(\frac{wv}{\lambda f}\right)$$

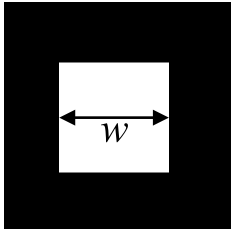
$$\mathrm{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$$

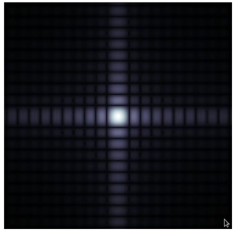


$$\frac{w\Delta x}{\lambda f} = 1$$

$$\Delta x = \frac{\lambda f}{w} = \frac{\lambda}{2\tan(\theta)} \approx \frac{\lambda}{2\sin(\theta)}$$



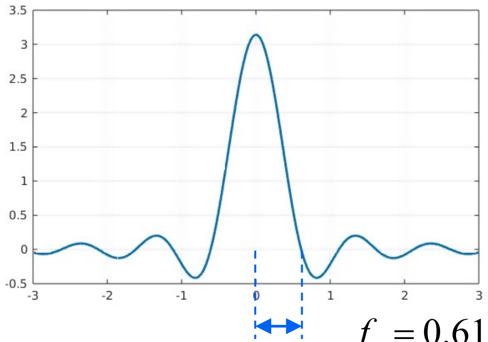




J. W. Goodman, Introduction to Fourier Optics, fourth edition. McMillan learning (2017)

The focused beam – circular aperture

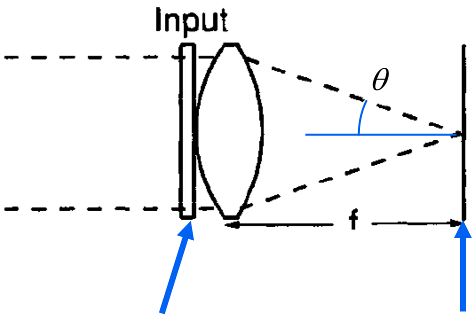
Circular pupil $$\mathcal{F}\{\mathrm{circ}(r)\} = 2\pi \frac{J_1(2\pi f_r)}{(2\pi f_r)} \qquad (2\text{-}36)$$



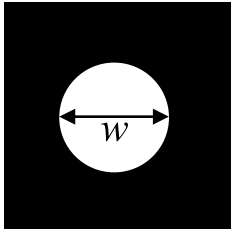
$\Delta\rho$ $\quad f_r = 0.61$

$$\frac{w}{2}\frac{\rho}{\lambda f} = 0.61$$

$$\Delta\rho = 1.22\frac{\lambda f}{w} = \frac{0.61\lambda}{\tan(\theta)} \approx \frac{0.61\lambda}{\sin(\theta)}$$



$P(x, y)$ $\quad U_F(u, v)$



J. W. Goodman, Introduction to Fourier Optics, fourth edition. McMillan learning (2017)

Many names for this one

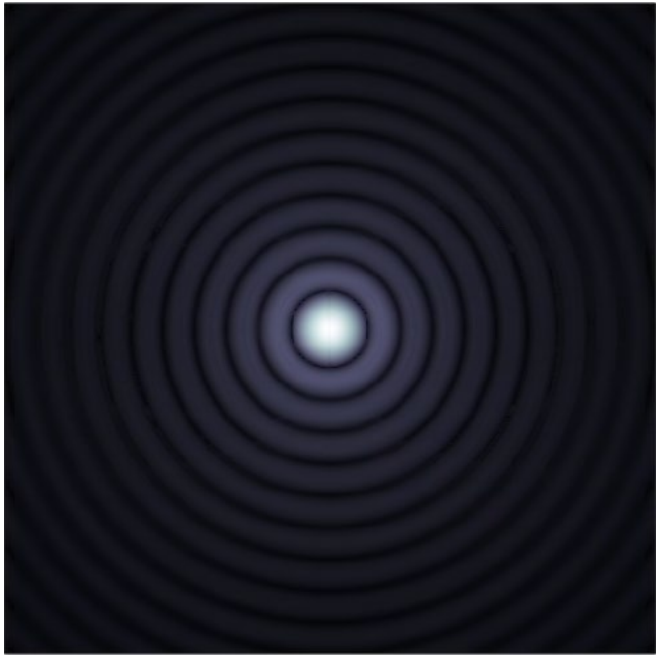
Besinc

Jinc

Sombrero function (?)

Airy disk (if squared)

$$h(\rho) = 2\pi \frac{J_1\left(2\pi \frac{w}{2} \frac{\rho}{\lambda f}\right)}{\left(2\pi \frac{w}{2} \frac{\rho}{\lambda f}\right)}$$



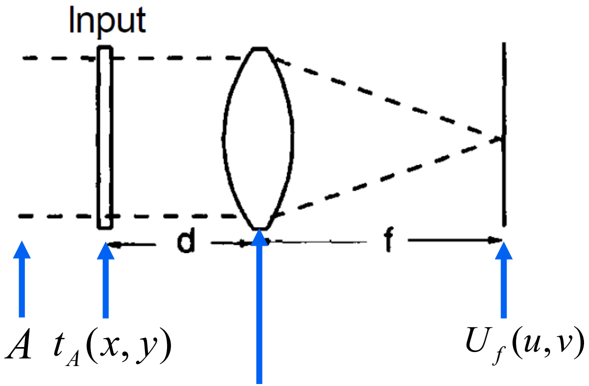
https://en.wikipedia.org/wiki/Sombrero_function

Fourier transforming properties of lenses

Fourier transform using a lens case (b)

$$U_f(u,v) = \mathcal{P}_f\left\{\mathcal{P}_d\left\{At_A(x,y)\right\}t_l(x,y)\right\}$$

$$U_l(x,y) = \mathcal{P}_d\left\{At_A(x,y)\right\} = \mathcal{F}^{-1}\left\{\mathcal{F}\left\{At_A(x,y)\right\}\exp\left(-i\pi\lambda d\left[f_x^2+f_y^2\right]\right)\right\}$$



$$t_l(x,y) = P(x,y)\exp\left\{-\frac{ik_0}{2f}(x^2+y^2)\right\}$$

$$U_f(u,v) = \frac{1}{i\lambda f}\exp\left\{\frac{ik_0}{2f}(u^2+v^2)\right\}\mathcal{F}\left\{U_l(x,y)P(x,y)\right\}\Bigg|_{\substack{f_x=\frac{u}{\lambda f}\\ f_y=\frac{v}{\lambda f}}}$$

$$U_f(x,y) = \frac{1}{i\lambda f}\exp\left\{\frac{ik_0}{2f}(u^2+v^2)\right\}\mathcal{F}\left\{At_A(x,y)\right\}\exp\left(-i\pi\lambda d\left[f_x^2+f_y^2\right]\right)\Bigg|_{\substack{f_x=\frac{u}{\lambda f}\\ f_y=\frac{v}{\lambda f}}}$$

J. W. Goodman, Introduction to Fourier Optics, fourth edition. McMillan learning (2017)

Fourier transform using a lens case (b)

$$U_f(x,y) = \frac{1}{i\lambda f} \exp\left\{\frac{ik_0}{2f}(u^2+v^2)\right\} \mathcal{F}\left\{At_A(x,y)\right\} \exp\left(-i\pi\lambda d\left[f_x^2+f_y^2\right]\right)\Big|_{\substack{f_x=\frac{u}{\lambda f} \\ f_y=\frac{v}{\lambda f}}}$$

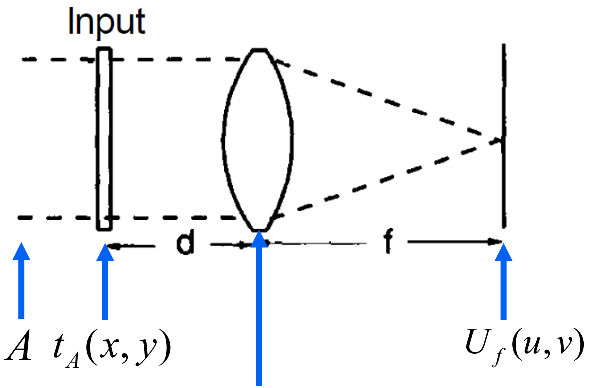
[redacted]

J. W. Goodman, Introduction to Fourier Optics, fourth edition. McMillan learning (2017)

Fourier transform using a lens case (b)

$$U_f(x,y) = \frac{1}{i\lambda f} \exp\left\{ \frac{ik_0}{2f}(u^2+v^2) \right\} \mathcal{F}\left\{ A t_A(x,y) \right\} \exp\left(-i\pi\lambda d \left[f_x^2 + f_y^2 \right] \right) \Bigg|_{\substack{f_x=\frac{u}{\lambda f} \\ f_x=\frac{u}{\lambda f}}}$$

$$U_f(x,y) = \frac{1}{i\lambda f} \exp\left\{ \frac{ik_0}{2f}(u^2+v^2)\left(1-\frac{d}{f}\right) \right\} \mathcal{F}\left\{ A t_A(x,y) \right\} \Bigg|_{\substack{f_x=\frac{u}{\lambda f} \\ f_y=\frac{v}{\lambda f}}}$$

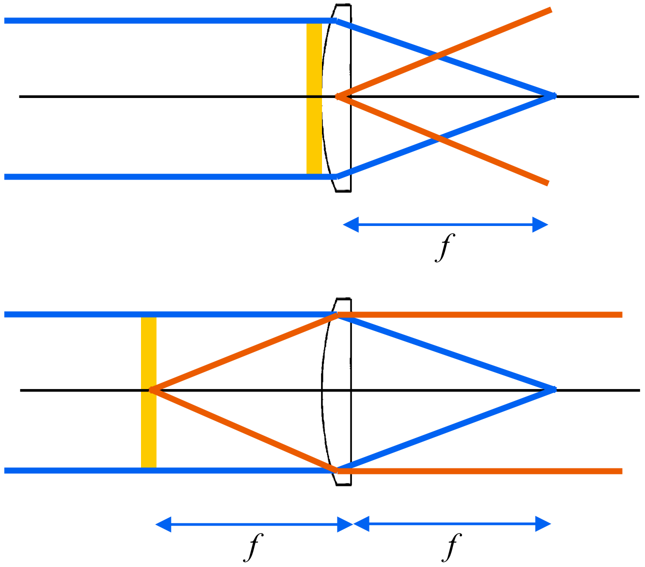


$$t_l(x,y) = P(x,y)\exp\left\{ -\frac{ik_0}{2f}(x^2+y^2) \right\}$$

J. W. Goodman, Introduction to Fourier Optics, fourth edition. McMillan learning (2017)

The external quadratic factor to the FT

$$U_f(x,y) = \frac{1}{i\lambda f}\exp\left\{\frac{ik_0}{2f}(u^2+v^2)\left(1-\frac{d}{f}\right)\right\}\mathcal{F}\left\{At_A(x,y)\right\}\Big|_{\substack{f_x=\frac{u}{\lambda f}\\ f_y=\frac{v}{\lambda f}}}$$



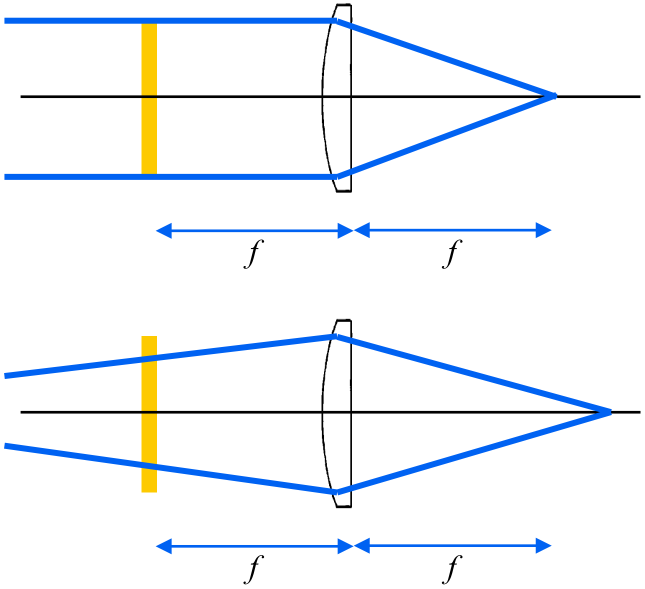
The quadratic phase corresponds to that obtained if the object was a point source

While it is not obvious without some further thought and analysis, our results show that *the quadratic-phase factor preceding the Fourier transform operation is always the quadratic-phase factor that would result at the transform plane from a point source of light located on the optical axis in the plane of the input transparency.*

J. W. Goodman, Introduction to Fourier Optics, fourth edition. McMillan learning (2017)

The location of the Fourier planes

$$U_f(x,y)=\frac{1}{i\lambda f}\exp\left\{\frac{ik_0}{2f}(u^2+v^2)\left(1-\frac{d}{f}\right)\right\}\mathcal{F}\{At_A(x,y)\}\Big|_{\substack{f_x=\frac{u}{\lambda f}\\ f_y=\frac{v}{\lambda f}}}$$



The Fourier transform plane need not be the focal plane of the lens performing the transform! Rather, *the Fourier transform always appears in the plane where the source is imaged.*

J. W. Goodman, Introduction to Fourier Optics, fourth edition. McMillan learning (2017)

Course contents

PSI
EPFL

Cover the unifying concepts of imaging, including with electrons, optics, X-rays.
In 2D, 3D, and beyond.

Chapter 2 - Fourier Optics and Imaging

- Lens transmissivity
- Fourier transforming properties of lenses
- Focused beams
- Imaging
- Frequency response of optical systems (coherent and incoherent cases)
- Wavefront aberrations
- Propagation through complicated paraxial optical systems

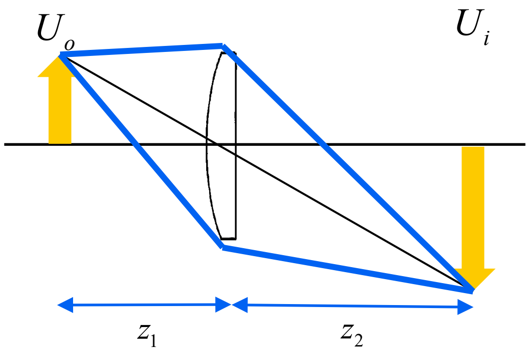
Imaging

What is imaging?

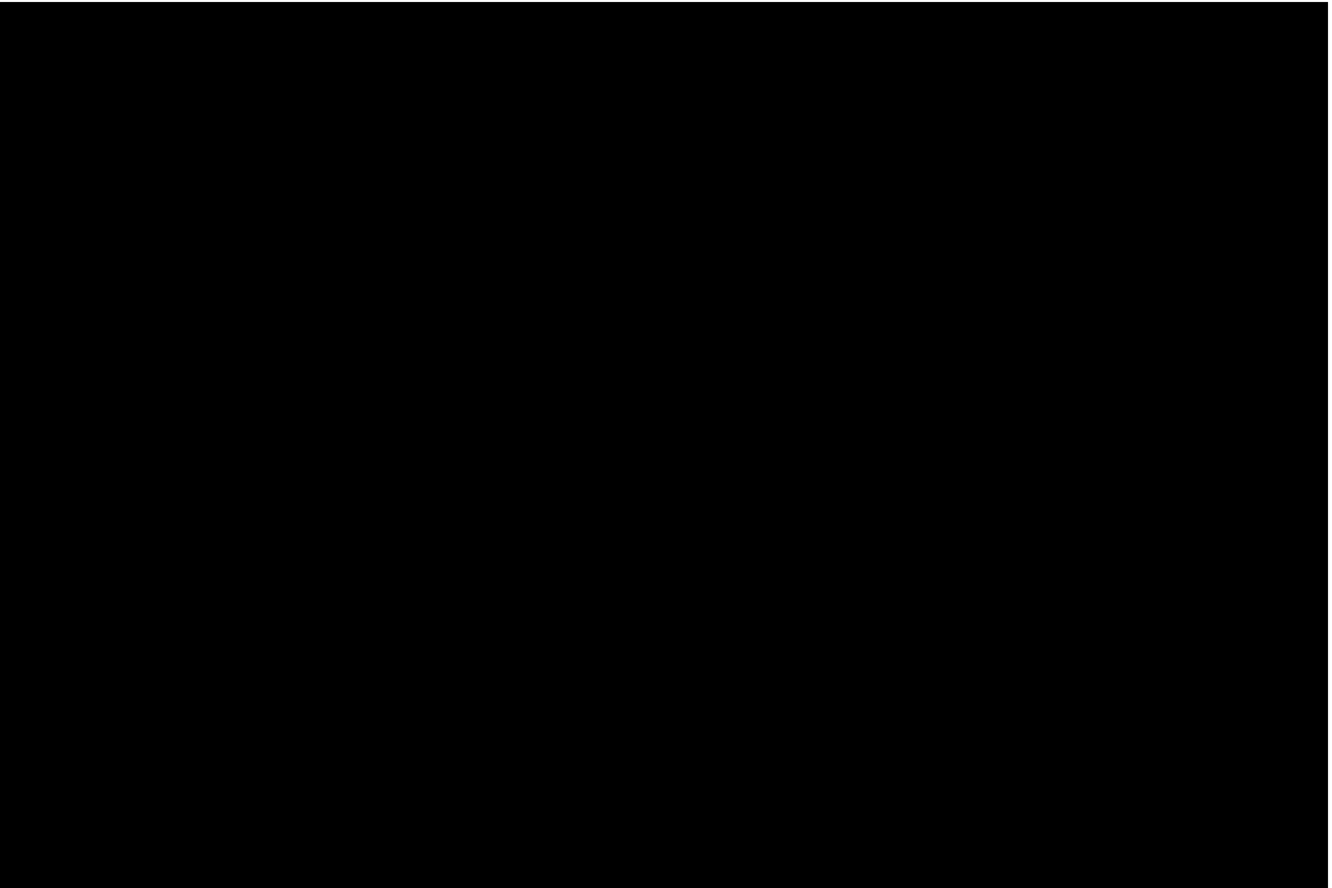
Mapping 1 to 1 between input and output plane

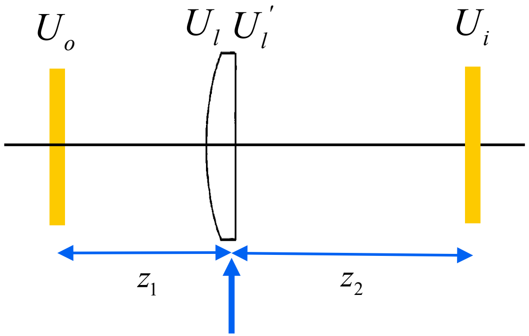
Some magnification is useful

There can be some distortion and/or blurring



Imaging





$$t_l(x,y) = P(x,y)\exp\left\{-\frac{ik_0}{2f}(x^2+y^2)\right\}$$

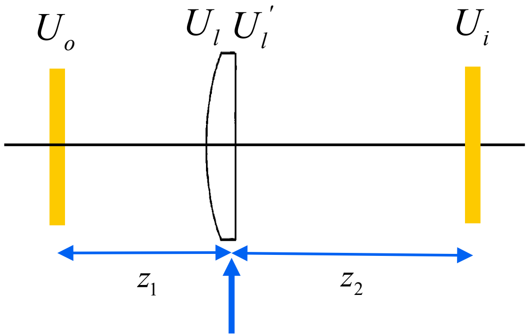
Note the derivation is different than Goodman's

Imaging

$$U_i(x,y)=\exp\left(\frac{ik}{2z_2}\left[x^2+y^2\right]\right)\left[\frac{1}{|M|}U_o\left(\frac{x}{M},\frac{y}{M}\right)\exp\left(\frac{ik}{2z_1M^2}\left[x^2+y^2\right]\right)\right]*h(x,y)$$

$$M=-\frac{z_2}{z_1}$$

$$h(x,y)=\frac{1}{M\lambda z_1 z_2}\mathcal{F}\left\{P(x_l,y_l)\right\}\Big|_{\substack{f_x=\frac{x}{\lambda z_2}\\ f_y=\frac{y}{\lambda z_2}}}$$



$$t_l(x,y)=P(x,y)\exp\left\{-\frac{ik_0}{2f}(x^2+y^2)\right\}$$

Object – Image are related by magnification and inversion

Effect of diffraction is convolution with an impulse response function *h(x,y)*

Impulse response can blur fine details, it is actually like a low-pass filter

For most practical applications the quadratic factors are not important – but for phase imaging they do appear

Note the derivation is different than Goodman's. I used direct propagation and integration, in the book you can find a description that starts with assumption that we will have an impulse response.

Frequency response of an optical system

Coherent case

Frequency response of an optical system

All diffraction effects can be associated with either of these two pupils

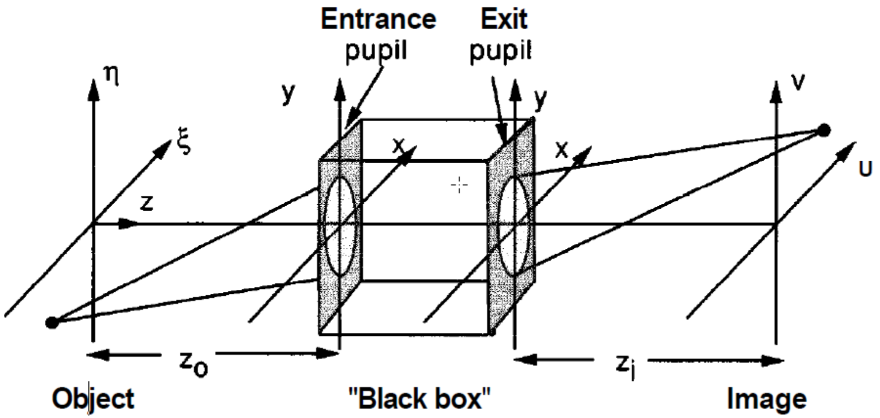
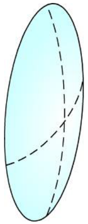
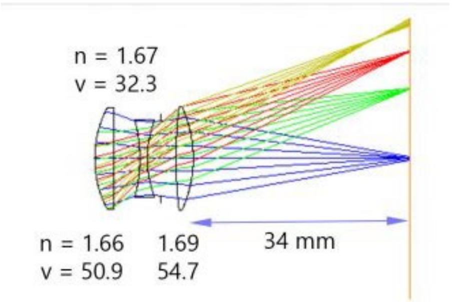


FIGURE 6.1
Generalized model of an imaging system.





J. W. Goodman, Introduction to Fourier Optics, fourth edition. McMillan learning (2017)

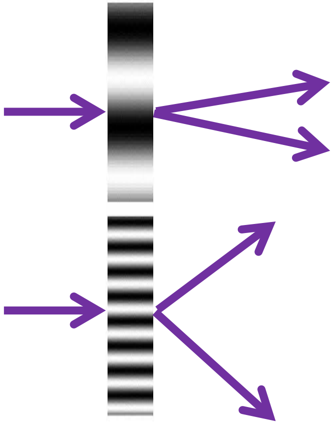
Frequency response of an optical system
Coherent case

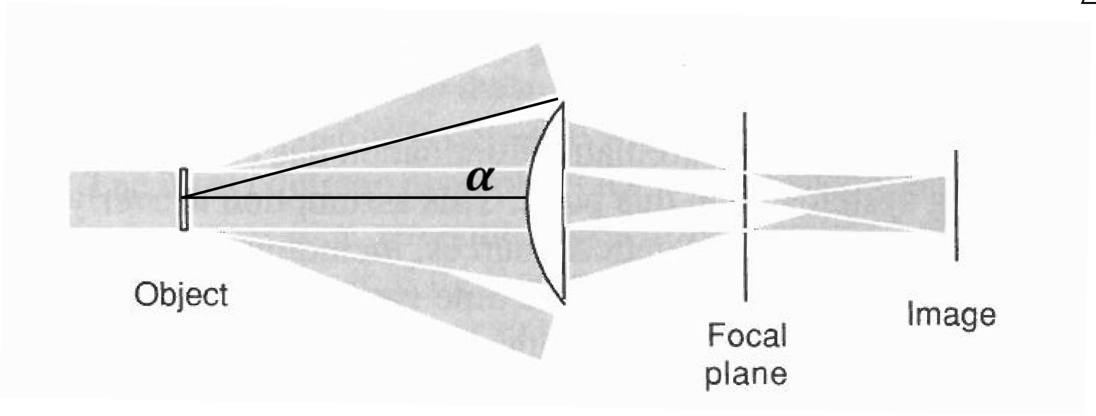
Abbe theory of image formation

High-resolution = high diffraction angles

Rayleigh resolution

$$\Delta x = \frac{0.61\lambda}{n \sin\alpha}$$





J. W. Goodman, Introduction to Fourier Optics, fourth edition. McMillan learning (2017)

Frequency response of an optical system
Coherent case

Real space

Coherent → when light appears to come from a single point

$$U_i(x,y) = \exp\left(\frac{ik}{2z_2}\left[x^2+y^2\right]\right)\left[\frac{1}{|M|}U_o\left(\frac{x}{M},\frac{y}{M}\right)\exp\left(\frac{ik}{2z_1M^2}\left[x^2+y^2\right]\right)\right]*h(x,y)$$

$$U_g(x,y) = \frac{1}{|M|}U_o\left(\frac{x}{M},\frac{y}{M}\right)\exp\left(\frac{ik}{2z_1M^2}\left[x^2+y^2\right]\right)$$

$$h(x,y) = \frac{1}{M\lambda z_1 z_2}\mathcal{F}\left\{P(x_l,y_l)\right\}_{\substack{f_x=\frac{x}{\lambda z_2}\\ f_y=\frac{y}{\lambda z_2}}}$$

$$U_i(x,y) = U_g(x,y) * h(x,y)$$

J. W. Goodman, Introduction to Fourier Optics, fourth edition. McMillan learning (2017)

Frequency response of an optical system
Coherent case

Real and reciprocal space

$$U_i(x,y) = U_g(x,y) * h(x,y)$$

$$H(f_x, f_y) = \mathcal{F}\{h(x,y)\} = \mathcal{F}\left\{\frac{1}{\lambda^2 z^2}\iint P(x,y)\exp\left(-i\frac{2\pi}{\lambda z_i}[ux+vy]\right)dxdy\right\} = P\left(-\lambda z_i f_x, -\lambda z_i f_y\right)$$

$$\mathcal{F}\{U_i(x,y)\} = \mathcal{F}\{U_g(x,y)\}\,P\left(-\lambda z_i f_x, -\lambda z_i f_y\right)$$

J. W. Goodman, Introduction to Fourier Optics, fourth edition. McMillan learning (2017)

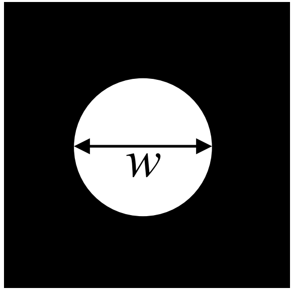
Circular pupil

Exit pupil diameter

$$P(x,y) = \mathrm{circ}\left(\frac{r}{w/2}\right)$$

$$H(x,y) = \mathrm{circ}\left(\frac{\lambda z_i f_r}{w/2}\right)$$

Coherent transfer function



$$f_c = \frac{1}{2}\frac{w}{\lambda z_i}$$

Cutoff frequency (w is exit pupil diameter)

J. W. Goodman, Introduction to Fourier Optics, fourth edition. McMillan learning (2017)

Frequency response of an optical system

Incoherent case

About coherence and incoherence

Statistical property of light

$$I = \left\langle \cos^2\left(\frac{2\pi x}{\tau} + \frac{\Delta\phi(t)}{2}\right)\right\rangle_t = \frac{1}{2} + \frac{1}{2}\left\langle \cos\left(\frac{2\pi x}{\tau/2} + \Delta\phi(t)\right)\right\rangle_t$$

If phase difference is constant fringe visibility will be 1

If phase difference is completely random fringe visibility will be 0

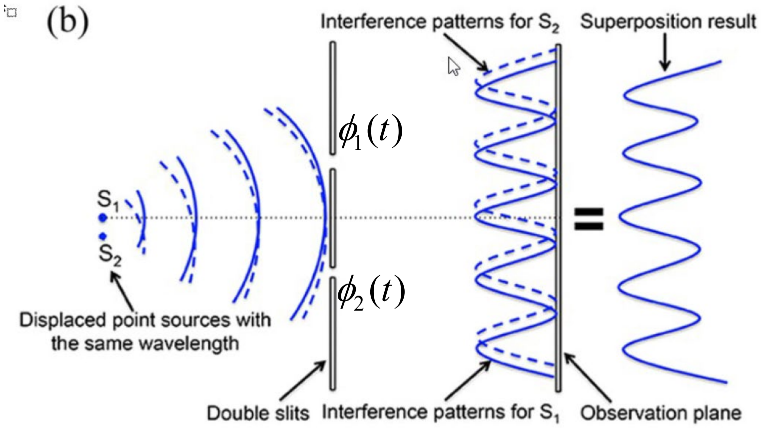
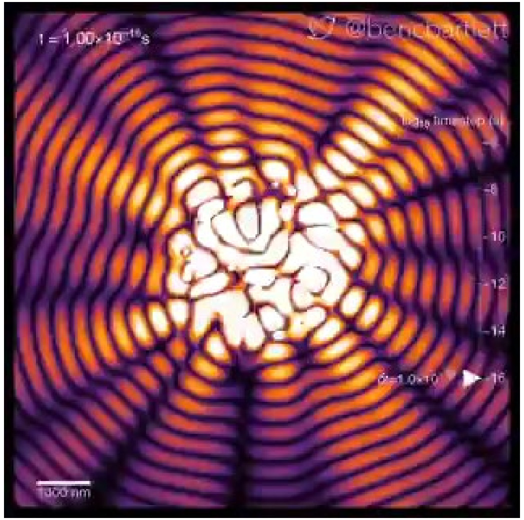


Image modified from https://www.nature.com/articles/s41598-017-06215-x

About coherence and incoherence

Thermal sources or vibrations much faster than the integration time



https://twitter.com/bencbartlett/status/1369396941730312193

Image intensity

For incoherent or partially coherent fields only intensity measurements averaged over time make sense

$$I_i(u,v) = \left\langle \left| U_i(u,v;t) \right|^2 \right\rangle$$

$$= \iint d\xi_1 d\eta_1 \iint d\xi_2 d\eta_2 h(u-\xi_1, v-\eta_1) h(u-\xi_2, v-\eta_2) \left\langle U_g(\xi_1,\eta_1;t-\tau_1) U^*_g(\xi_2,\eta_2;t-\tau_2) \right\rangle$$

Taking into account a small-area h(u,v) over which the time differences are small we can drop the time-delay difference

$$I_i(u,v) = \iint d\xi_1 d\eta_1 \iint d\xi_2 d\eta_2 h(u-\xi_1, v-\eta_1) h(u-\xi_2, v-\eta_2) J_g(\xi_1,\eta_1;\xi_2,\eta_2)$$

$$J_g(\xi_1,\eta_1;\xi_2,\eta_2) = \left\langle U_g(\xi_1,\eta_1;t) U^*_g(\xi_2,\eta_2;t) \right\rangle$$

= **mutual intensity** = measure of spatial coherence

Incoherent imaging

Fields uncorrelated (for distances greater than λ)

$$I_i(u,v) = \iint d\xi_1 d\eta_1 \iint d\xi_2 d\eta_2 h(u-\xi_1, v-\eta_1) h(u-\xi_2, v-\eta_2) J_g(\xi_1,\eta_1;\xi_2,\eta_2)$$

$$J_g(\xi_1,\eta_1;\xi_2,\eta_2) = \langle U_g(\xi_1,\eta_1;t) U^*_g(\xi_2,\eta_2;t)\rangle$$

$$\langle U_g(\tilde{\xi}_1,\tilde{\eta}_1;t) U^*_g(\tilde{\xi}_2,\tilde{\eta}_2;t)\rangle = \kappa I_g(\tilde{\xi}_1,\tilde{\eta}_1)\delta(\tilde{\xi}_1-\tilde{\xi}_2, \tilde{\eta}_1-\tilde{\eta}_2)$$

$$I_i(u,v) = \kappa \iint I_g(\xi,\eta) |h(u-\xi, v-\eta)|^2 \, d\xi d\eta$$

Take square magnitude and convolve (incoherently)

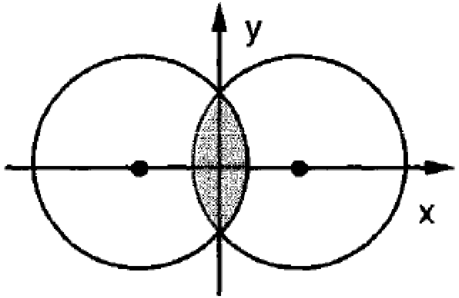
Names:

Incoherent impulse response

Intensity impulse response

Point-spread function (PSF)

Incoherent imaging



$$I_i(u,v) = \kappa \iint I_g(\xi,\eta) \left| h(u-\xi, v-\eta) \right|^2 d\xi d\eta$$

$$I_i(u,v) = \kappa \mathcal{F}^{-1}\left\{ \mathcal{F}\left\{ I_g(\xi,\eta) \right\} OTF\left(f_x, f_y\right) \right\}$$

$$OTF\left(f_x, f_y\right) = P\left(-\lambda z_i f_x, -\lambda z_i f_y\right) \bigstar P\left(-\lambda z_i f_x, -\lambda z_i f_y\right)$$ Optical transfer function

$$MTF\left(f_x, f_y\right) = \frac{\left| OTF\left(f_x, f_y\right) \right|}{\left| OTF(0,0) \right|}$$ Modulation transfer function

Autocorrelation $\mathcal{F}\{|g(x,y)|^2\} = G\left(f_x, f_y\right) * G^*\left(-f_x, -f_y\right) = G\left(f_x, f_y\right) \bigstar G\left(f_x, f_y\right)$

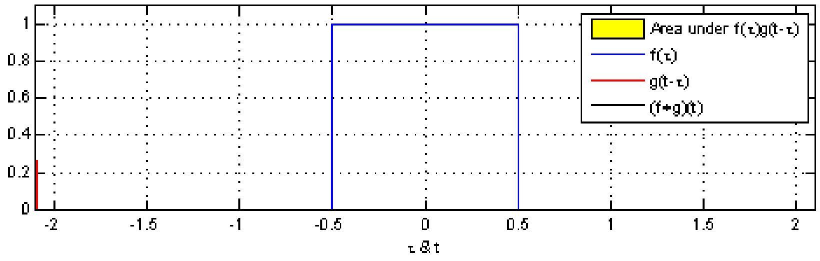
J. W. Goodman, Introduction to Fourier Optics, fourth edition. McMillan learning (2017)

Theorems of the FT

Convolution $$g(x,y) * h(x,y) = \iint_{-\infty}^{\infty} g(\xi,\eta)h(x-\xi,y-\eta)d\xi d\eta$$

$$\mathcal{F}\{g(x,y) * h(x,y)\} = G(f_x,f_y)H(f_x,f_y)$$

Sifting property $$g(x,y) * \delta(x-a,y-b) = g\,(x-a,y-b)$$



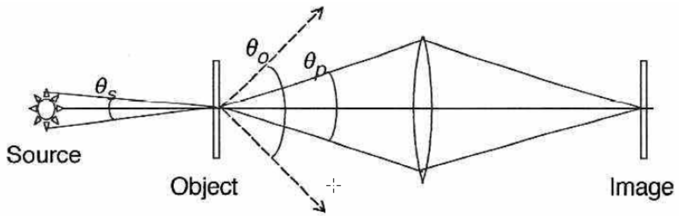
J. W. Goodman, Introduction to Fourier Optics, fourth edition. McMillan learning (2017)
https://en.wikipedia.org/wiki/Convolution#/media/File:Convolution_of_box_signal_with_itself2.gif
Matlab "imfilter.m" documentation

Coherent vs incoherent imaging

Coherent

$$I_i(u,v) = \left| \iint U_g(\xi,\eta)\, h(u-\xi, v-\eta)\, d\xi d\eta \right|^2$$

$$U_i(x,y) = \mathcal{F}^{-1}\left\{ \mathcal{F}\left\{U_g(x,y)\right\} P\left(-\lambda z_i f_x, -\lambda z_i f_y\right)\right\}$$



Coherent if $\theta_s << \theta_p$

Incoherent if $\theta_s \geq \theta_o + \theta_p$

Partially coherent in-between

Incoherent

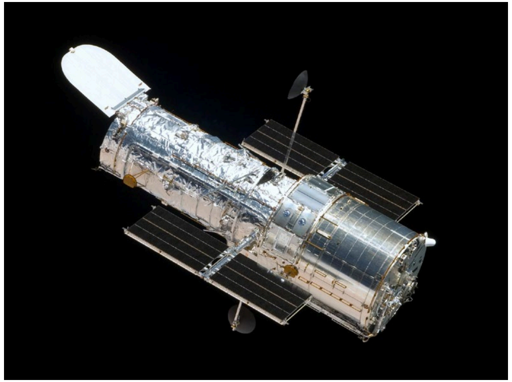
$$I_i(u,v) = \kappa \iint I_g(\xi,\eta) \left| h(u-\xi, v-\eta) \right|^2 d\xi d\eta$$

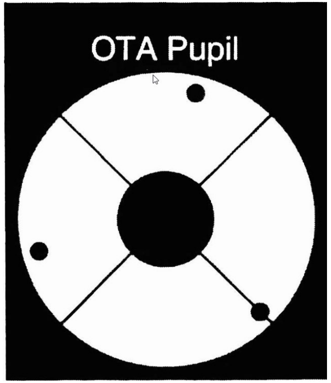
$$I_i(u,v) = \kappa \mathcal{F}^{-1}\left\{ \mathcal{F}\left\{ I_g(\xi,\eta)\right\} OTF\left(f_x, f_y\right)\right\}$$

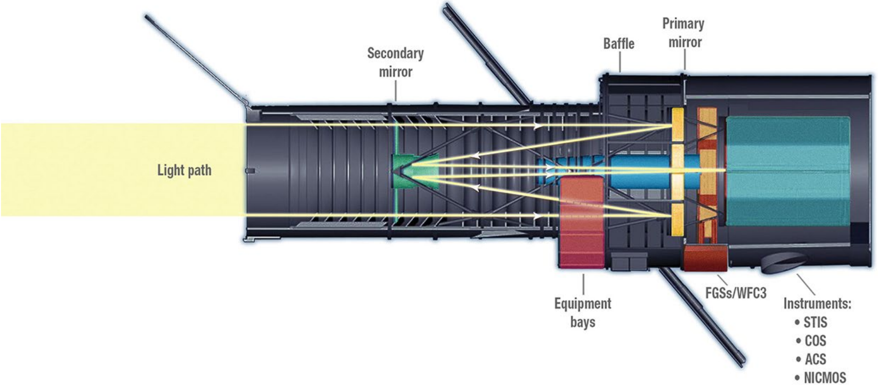
$$OTF\left(f_x, f_y\right) = P\left(-\lambda z_i f_x, -\lambda z_i f_y\right) \bigstar P\left(-\lambda z_i f_x, -\lambda z_i f_y\right)$$

Telescope pupils

Hubble space telescope



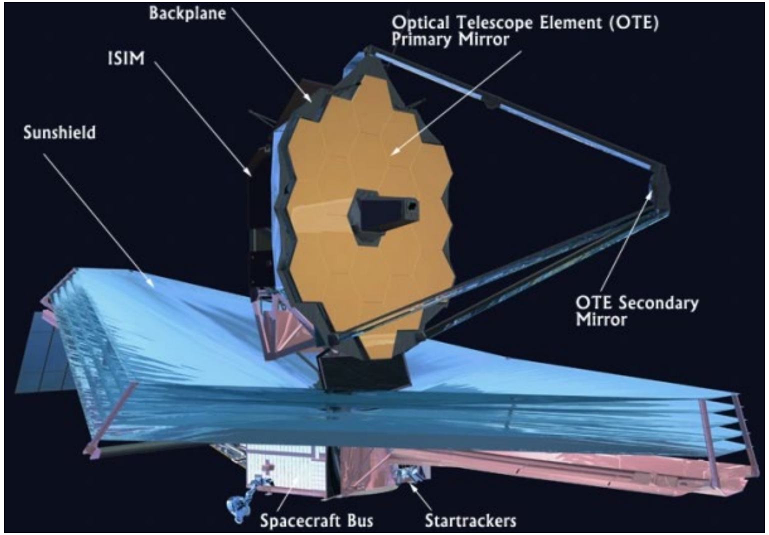




https://en.wikipedia.org/wiki/Hubble_Space_Telescope
https://hubblesite.org/contents/media/images/4520-Image.html?Tag=Hubble%20Mission
M. Robberto et al. «Performance of HST as an infrared telescope» SPIE Proc. **4013**, UV, Optical, and IR Space Telescopes and Instruments (2000)

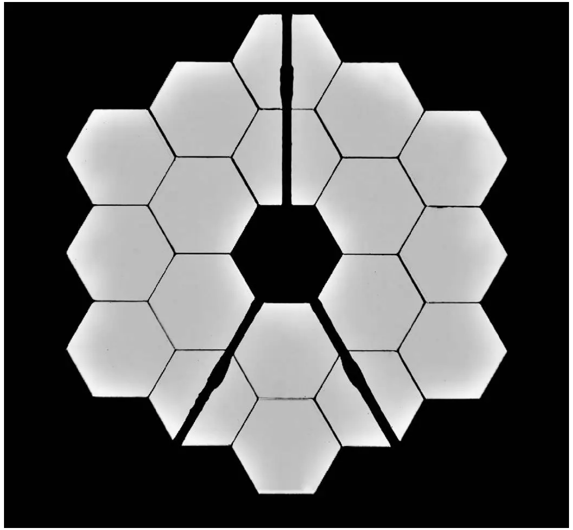
Telescope pupils

JWST



https://spacenews.com/jwst-undamaged-from-payload-processing-incident/
https://astrobites.org/guides/guide-to-major-telescopes/the-james-webb-space-telescope/

The JWST point spread function



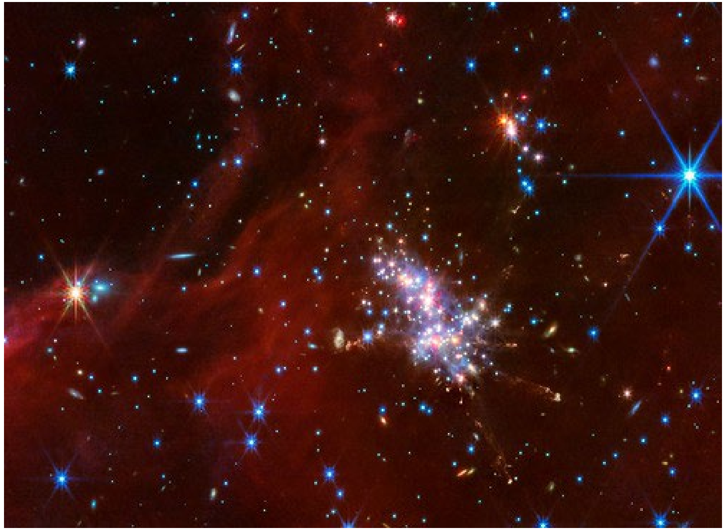
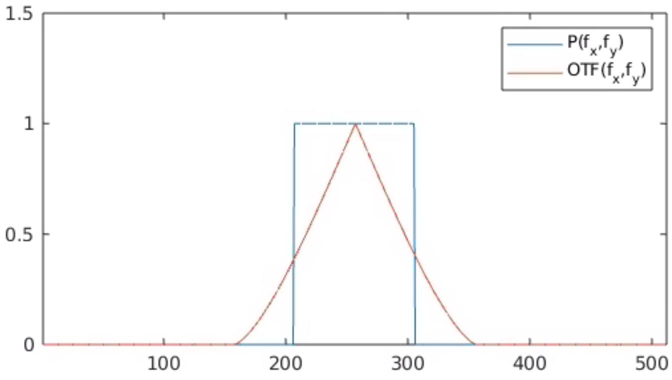
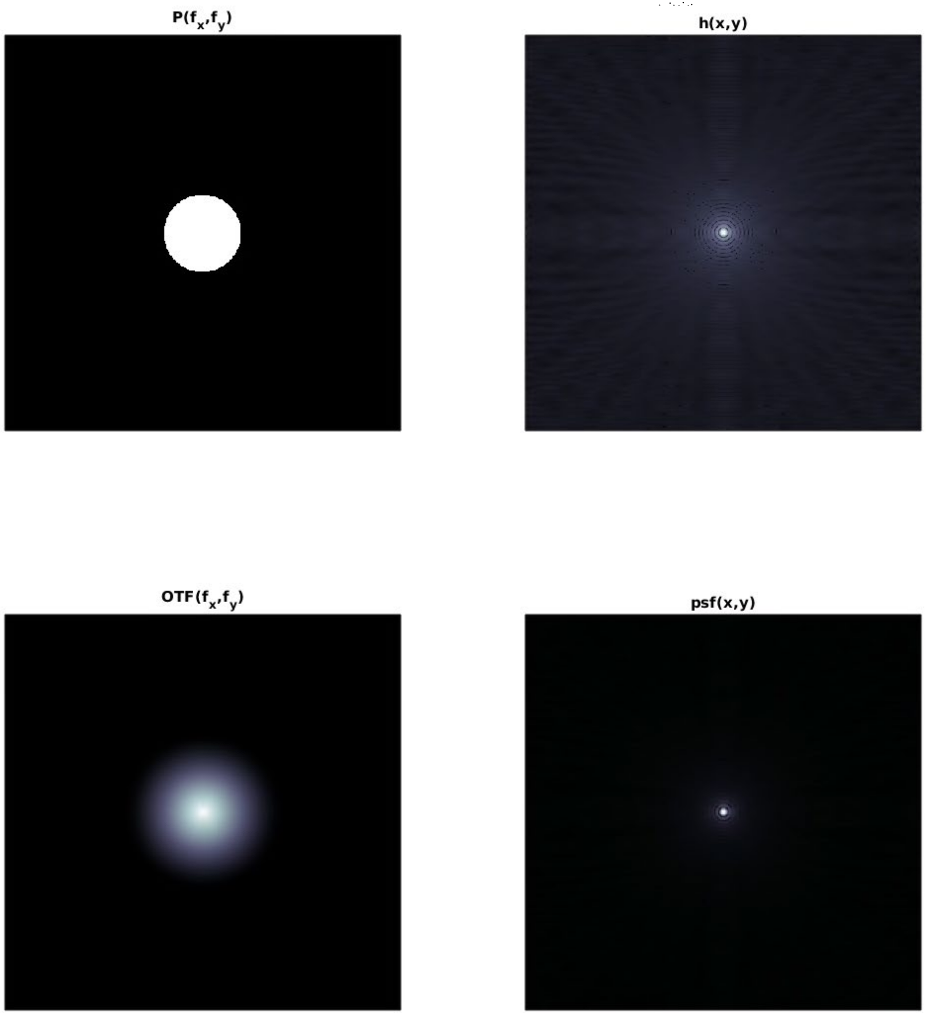


Image from https://www.nasa.gov/image-article/mirrors-aligned-webb-telescopes-first-full-color-images-due-july/
https://science.nasa.gov/mission/webb/#Featured-Image

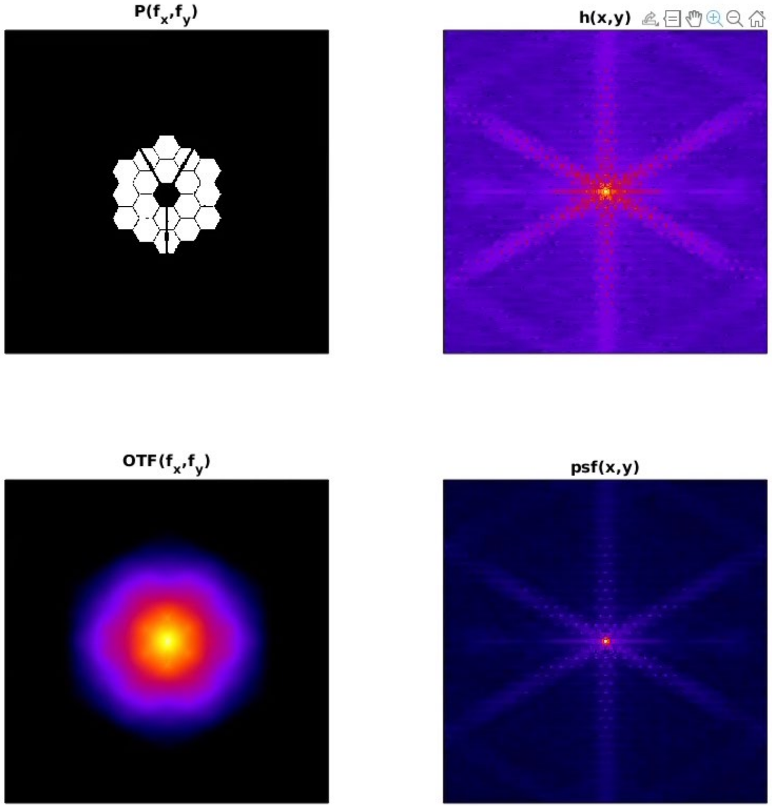
Demonstration of different types of PSFs

OTF_psf.m





The JWST point spread function



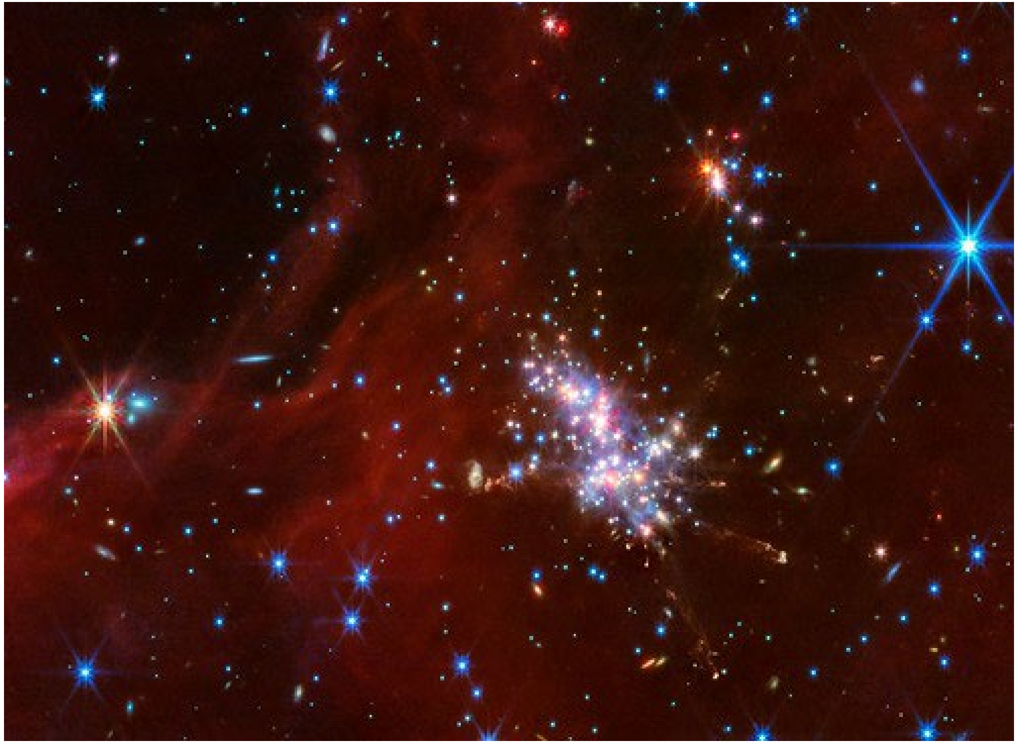
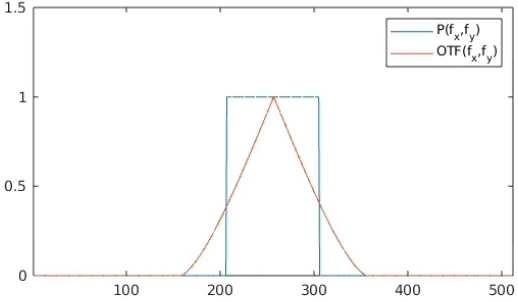
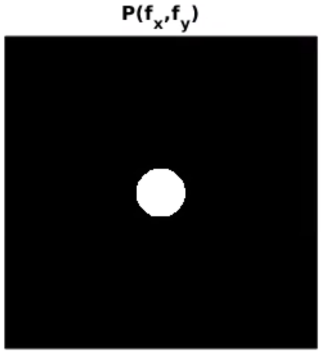


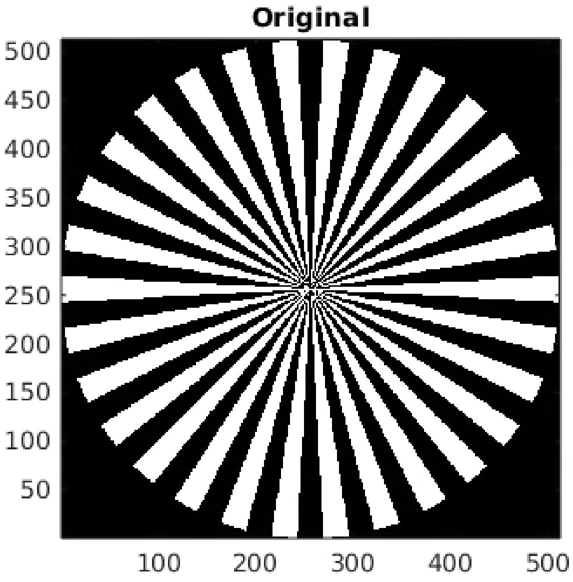
Image from https://www.nasa.gov/image-article/mirrors-aligned-webb-telescopes-first-full-color-images-due-july/
https://science.nasa.gov/mission/webb/#Featured-Image

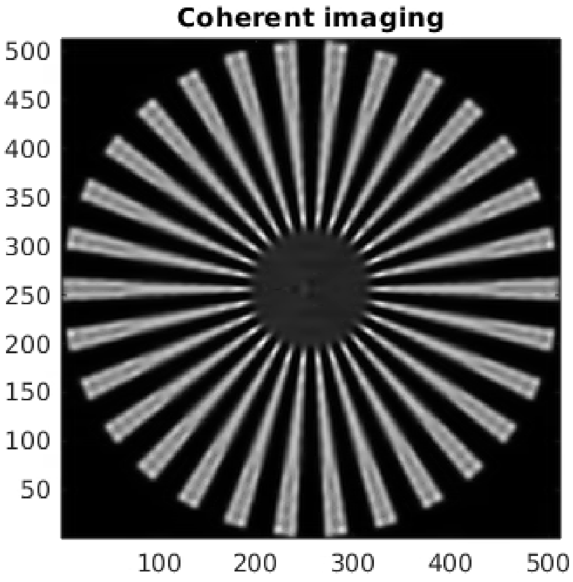
Coherent vs incoherent imaging

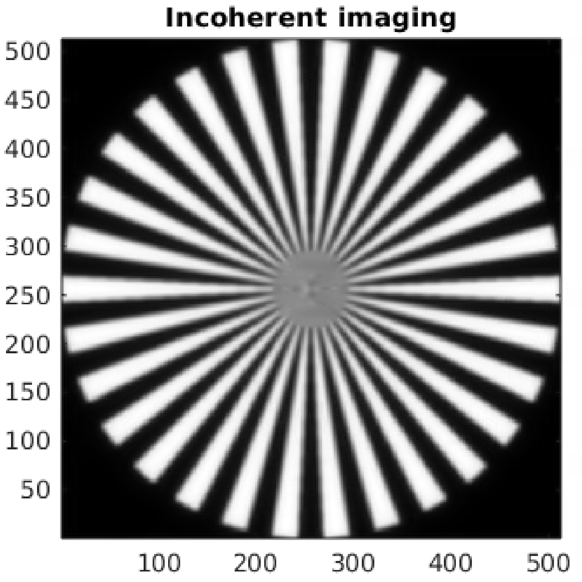
OTF_psf.m
(with images)



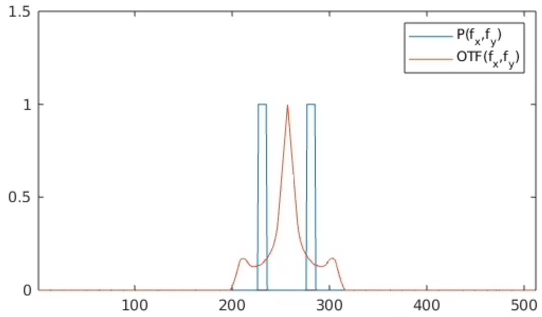


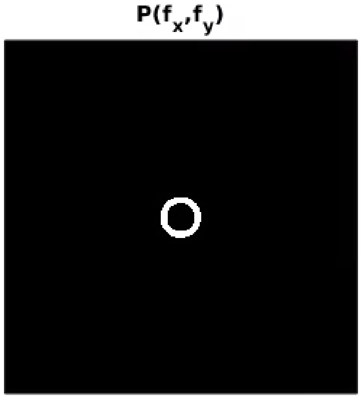






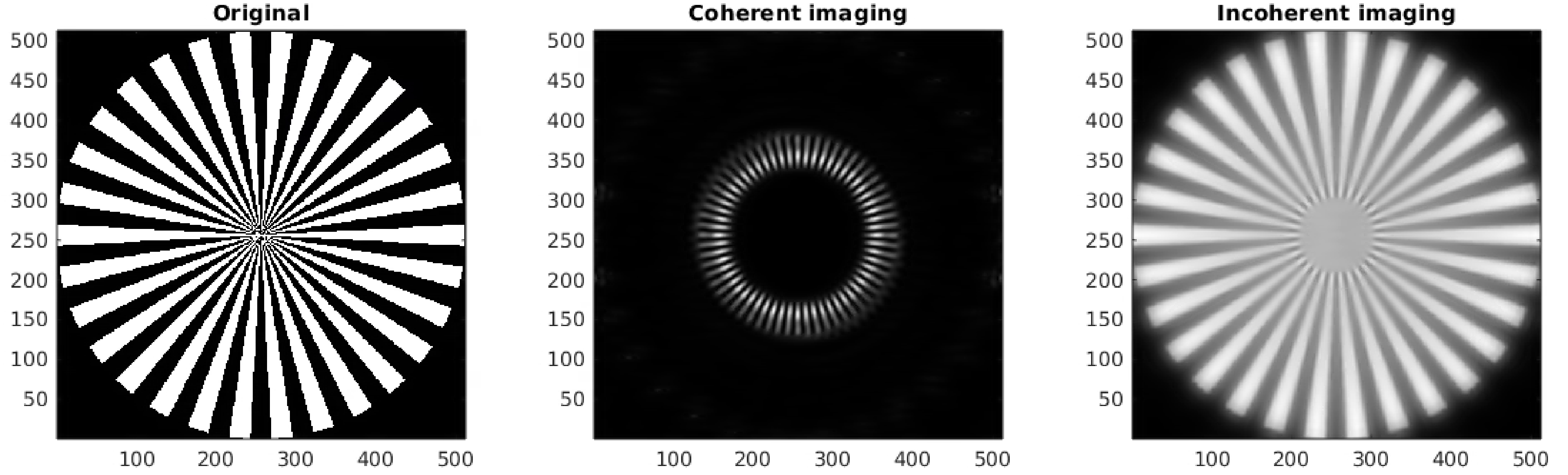
Coherent vs incoherent imaging





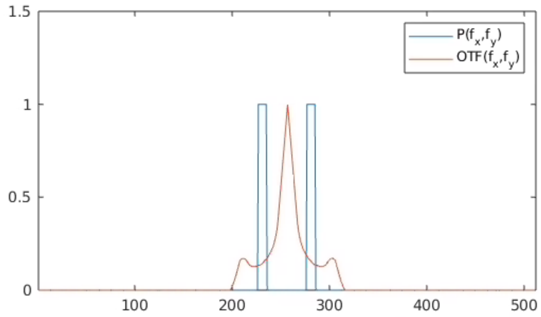
OTF_psf.m

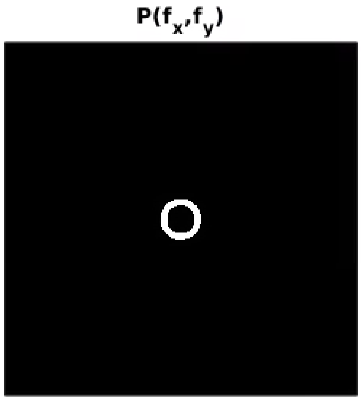
(with images)

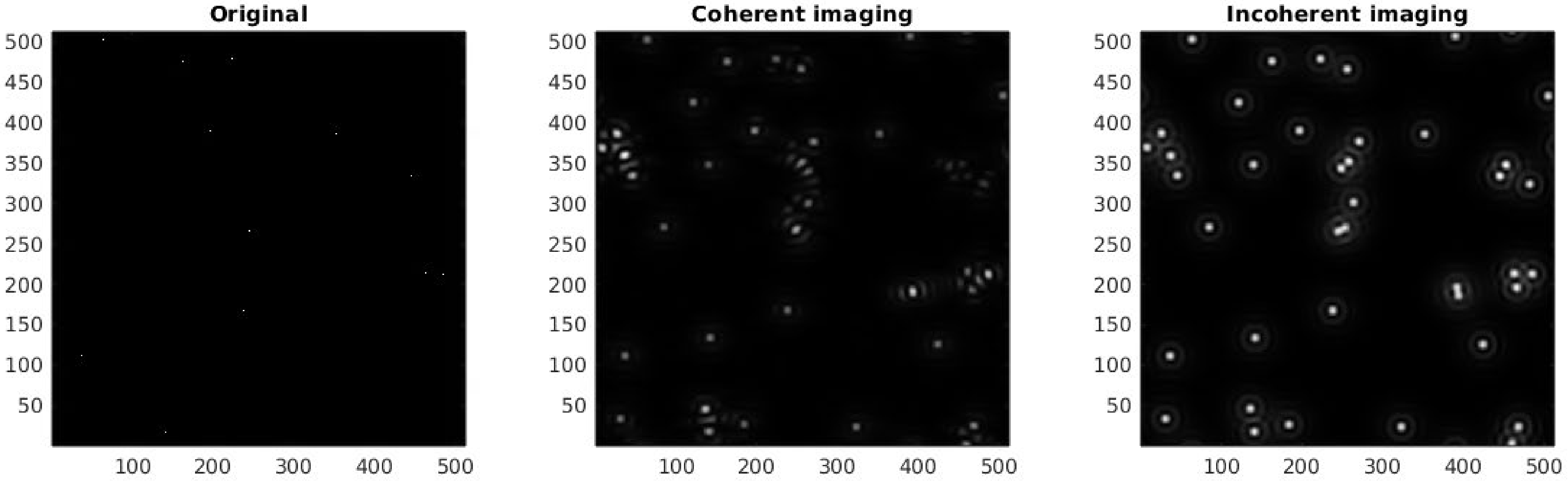


Coherent vs incoherent imaging

OTF_psf.m
(with images)







Wavefront aberrations

Wavefront aberrations

Perfect spherical wavefront → *diffraction limited*

Wavefront aberrations → Deviation from perfect spherical wavefront

→ Phase error in the exit pupil

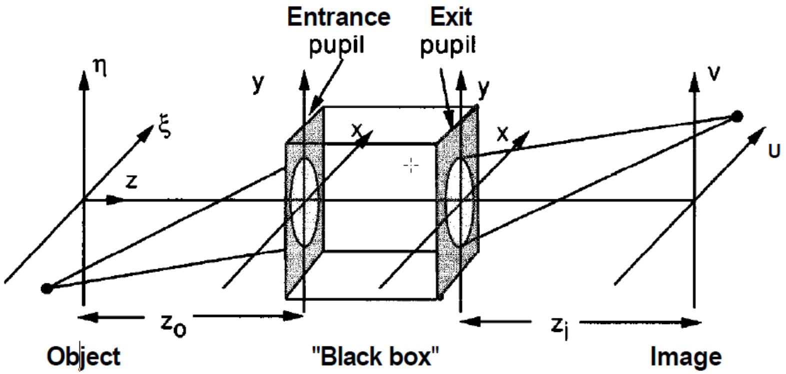
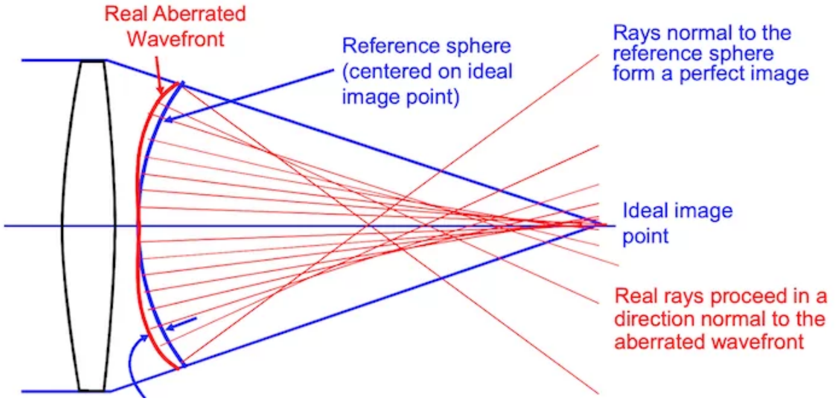
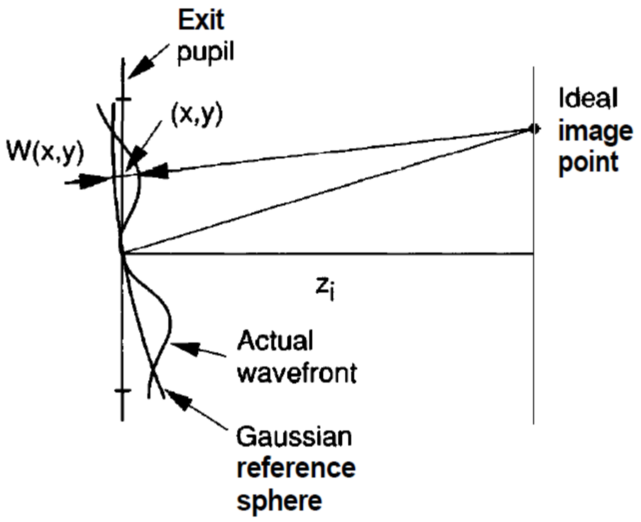


FIGURE 6.1
Generalized model of an imaging system.



J. W. Goodman, Introduction to Fourier Optics, fourth edition. McMillan learning (2017)
https://www.opticsforhire.com/blog/rms-wavefront-error-explained/

Wavefront aberrations



Aberrations due to

- Optical design
- Manufacturing errors
- Aberrating media (e.g. turbulence)
- Thermal stresses
- etc

Generalized pupil function (complex valued)

$$\hat{P}(x,y) = P(x,y)\exp\left[ikW(x,y)\right]$$

Coherent transfer function

$$H(f_x, f_y) = \hat{P}(\lambda z_i f_x, \lambda z_i f_y)$$

J. W. Goodman, Introduction to Fourier Optics, fourth edition. McMillan learning (2017)

Wavefront aberrations

Always reduces contrast on the OTF → for each frequency

Can cause contrast reversal

$$OTF\left(f_x, f_y\right) = P\left(-\lambda z_i f_x, -\lambda z_i f_y\right) \bigstar P\left(-\lambda z_i f_x, -\lambda z_i f_y\right)$$

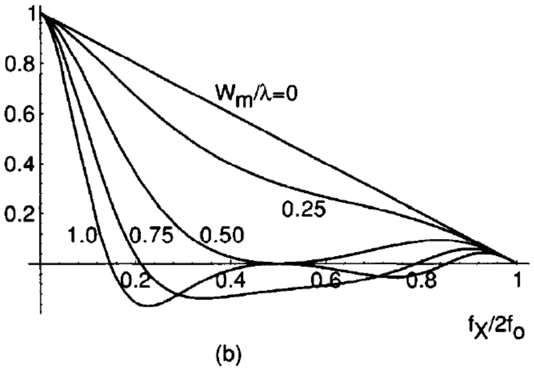


FIGURE 6.11
OTF for a focusing error in a system with a square pupil. (a) Three-dimensional plot with $f_X/2f_o$ along one axis and W_m/λ along the other axis. (b) Cross section along the f_X axis with W_m/λ as a parameter.

J. W. Goodman, Introduction to Fourier Optics, fourth edition. McMillan learning (2017)

Defocus aberration

Gradual attenuation of contrast wrt frequency

OTF = 0   those frequencies don't make it through

OTF < 0   contrast reversal

Maximum number of waves of OPD W_m / λ

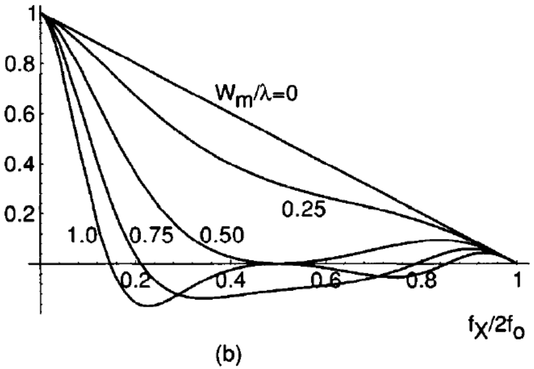
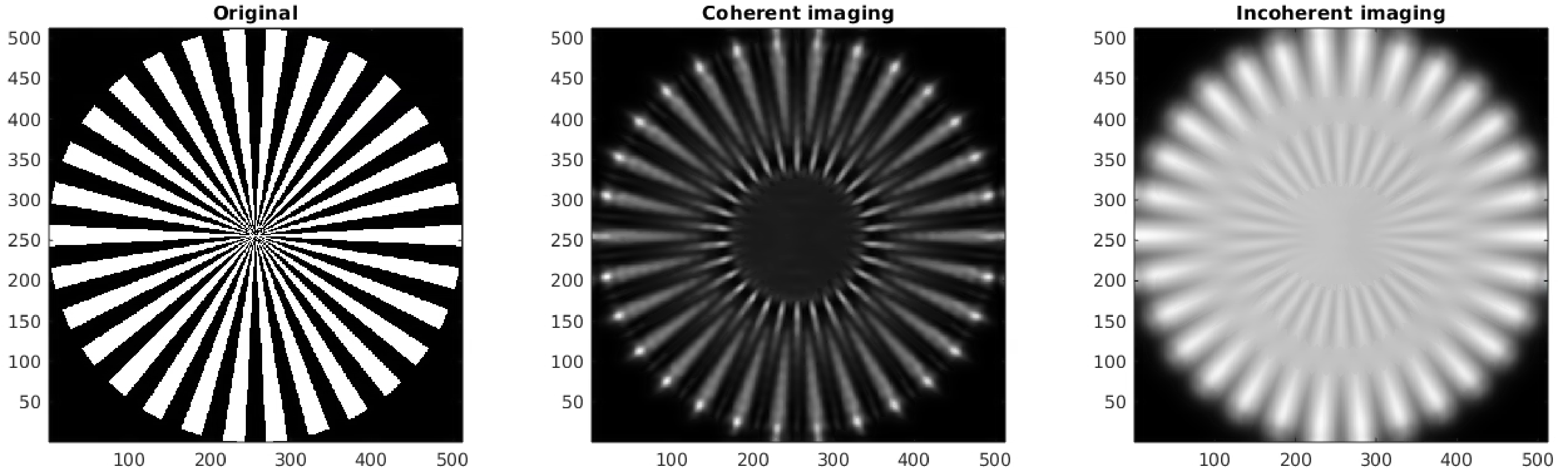
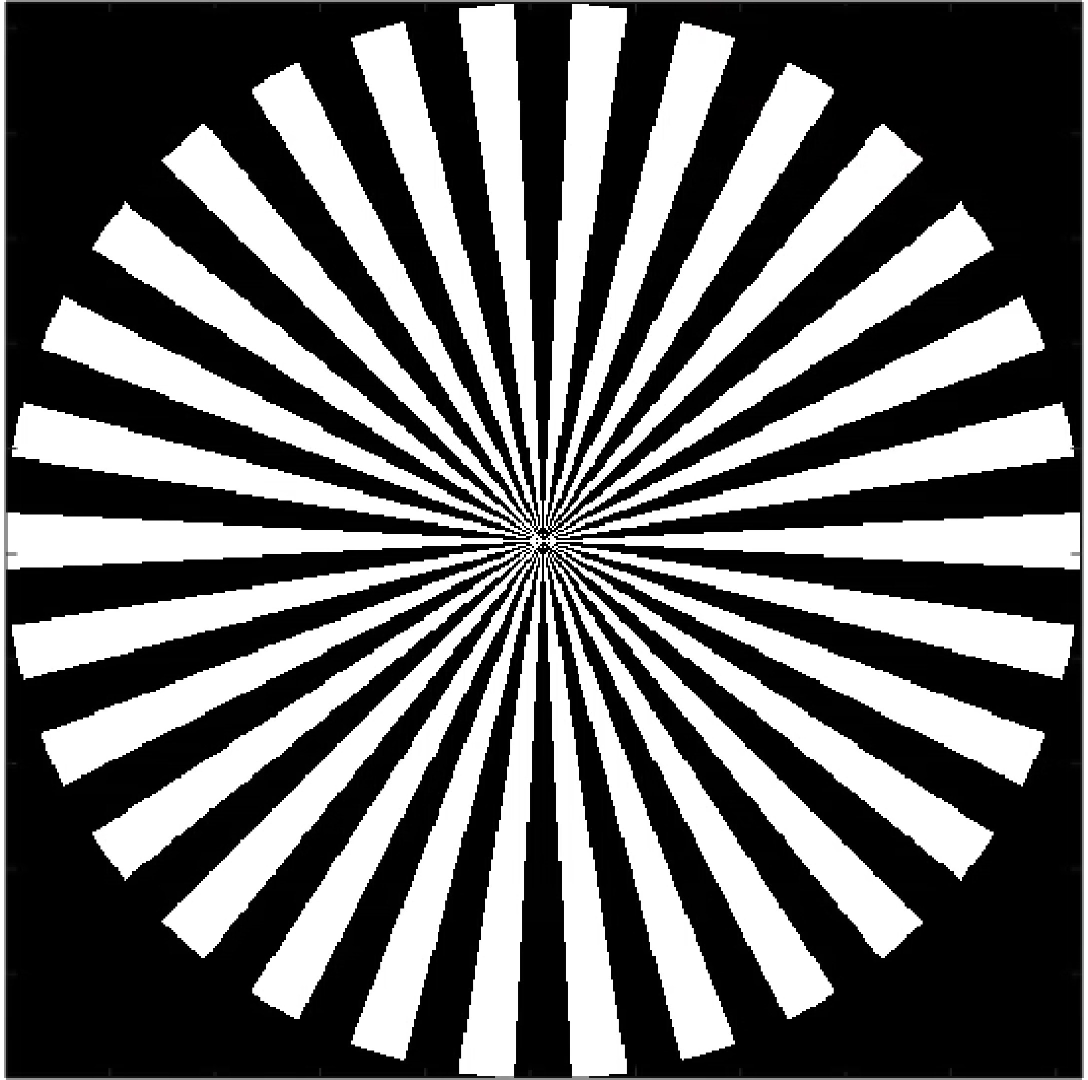


FIGURE 6.11
OTF for a focusing error in a system with a square pupil. (a) Three-dimensional plot with $f_X/2f_o$ along one axis and W_m/λ along the other axis. (b) Cross section along the f_X axis with W_m/λ as a parameter.

Showing defocus of a Siemens star to see contrast reversal

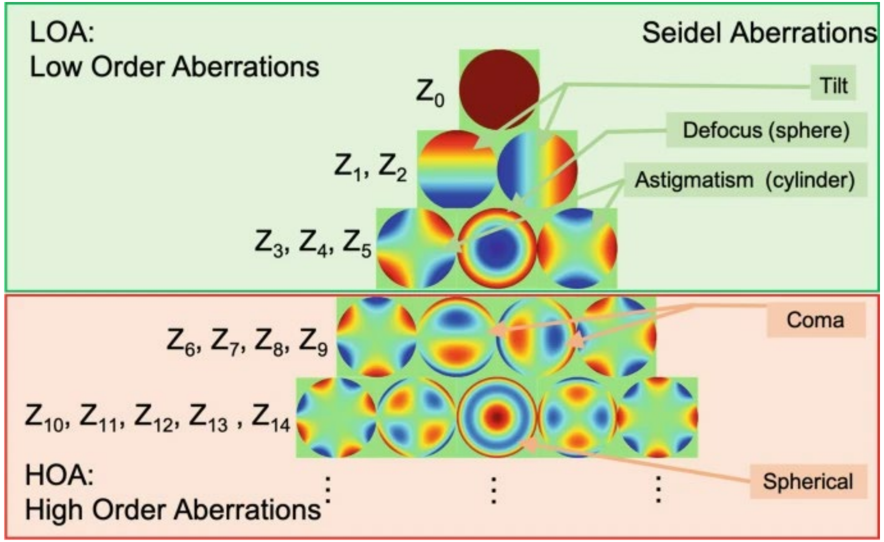
```
%%% Pupil %%%
masktype = 'circle'; % circle, ring, jwst
Nmask = 70;    % Circle diameter
ring_ratio = 1.5; % R1/R2 for ring
N = 512;

%%% Image %%%
image_type = 'spokes'; % spokes, stars, cameraman
    num_spokes = 15;
    num_stars = 40;
show_image = true;

%%% Aberrations %%%
defocus = 0.9; % wave
show_aberrations = 1;
```





Other aberrations

Seidel aberrations



Atmospheric turbulence

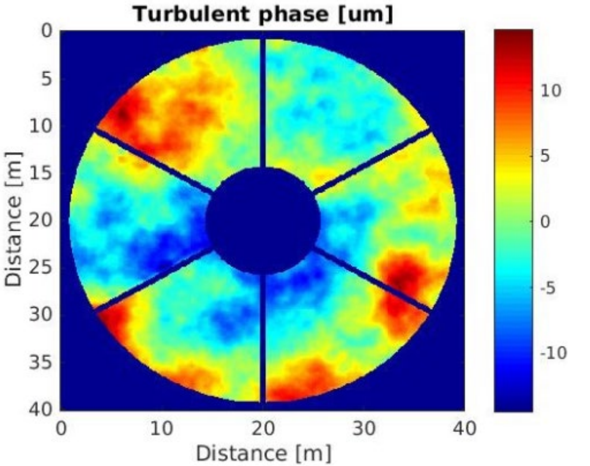


Figure 4: Example of atmospheric phase (in μm) with an average piston over the pupil set to 0. Segments are labelled from the far right and anti-clockwise.

10.26698/AO4ELT5.0015
https://link.springer.com/chapter/10.1007/978-3-031-50666-6_4

Diffraction limited

Imaging performance limited by diffraction, not by aberrations

Maréchal criterion. Formally, that a wavefront can be regarded as diffraction-limited if its RMS phase error is 14 times less than its wavelength.

$$\sigma_W < \frac{\lambda}{14} \approx 0.071\lambda$$

Strehl ratio. PSF peak intensity / ideal PSF peak intensity

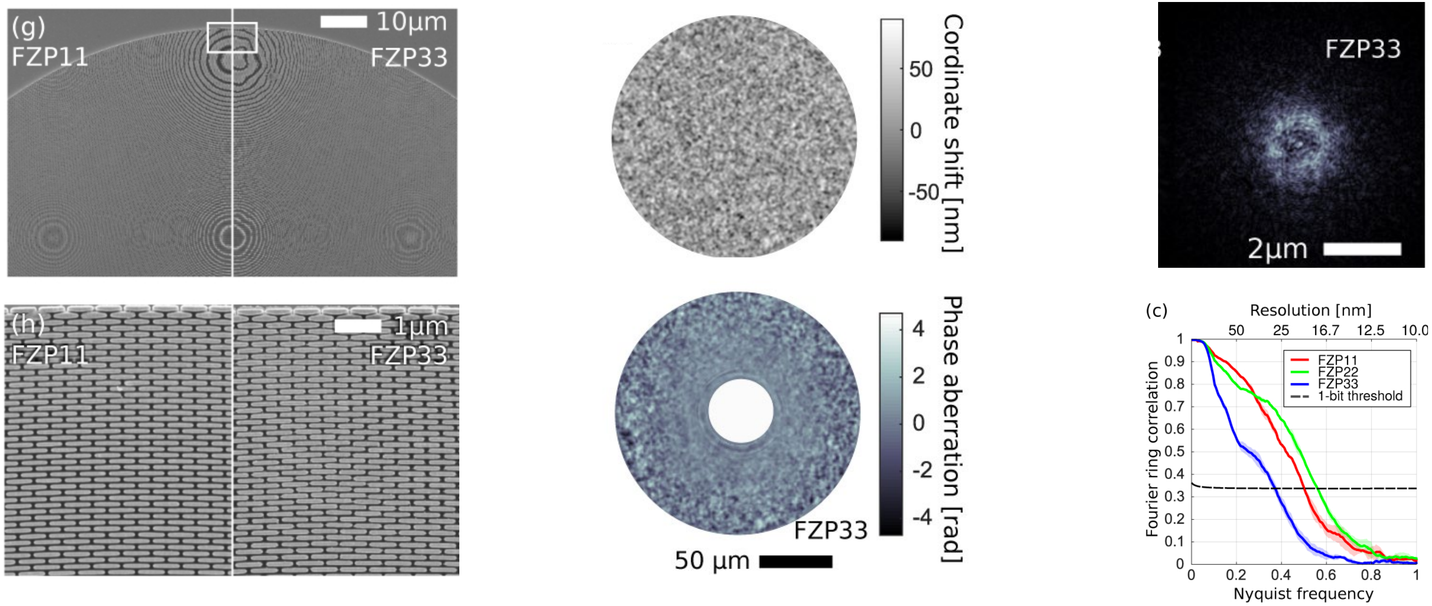
S > 0.8 is considered diffraction limited

$$S = \frac{|h_{\text{max}}|^2}{|h_{\text{perfect,max}}|^2} = \left|\left\langle \exp(ikW) \right\rangle\right|^2 \approx \exp\left(-k^2\sigma_W^2\right)$$

https://en.wikipedia.org/wiki/Andr%C3%A9_Mar%C3%A9chal
https://en.wikipedia.org/wiki/Strehl_ratio

Ugly lenses – engineered distortions for ptychography

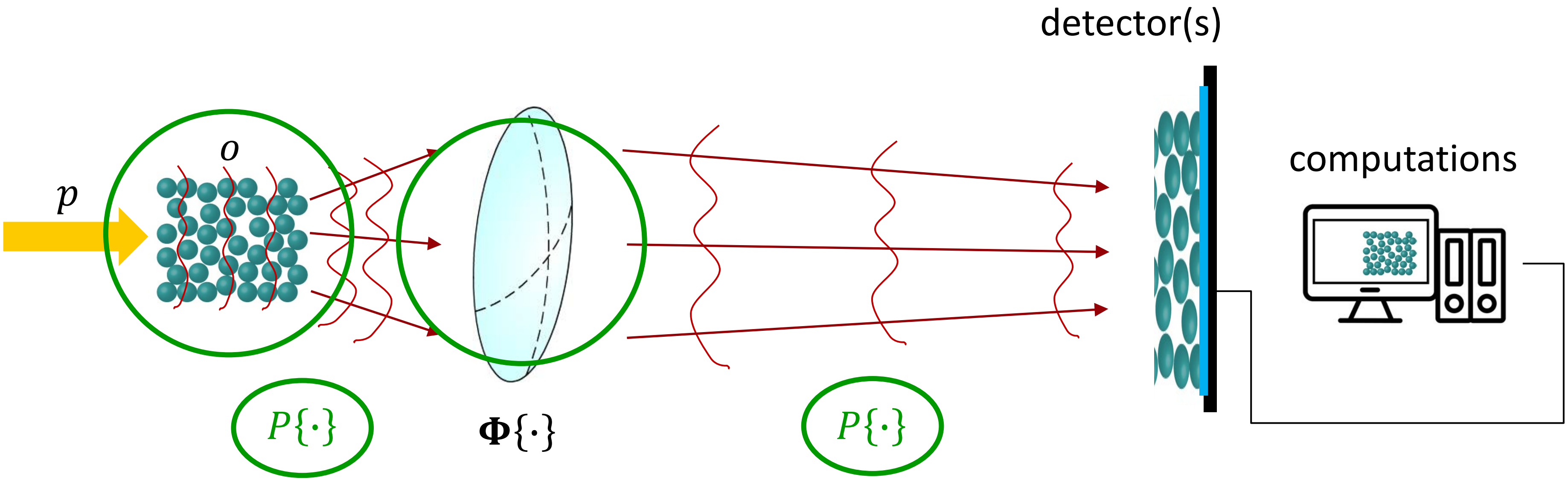
We introduce engineered wavefront distortions to introduce **structure** in the beam and increase diversity in the datasets



M. Odstrčil, M. Lebugle, M. Guizar-Sicairos, C. David, and M. Holler, “Towards optimized illumination for high-resolution ptychography,“ Opt. Express **27**, 14981 (2019)

Lenses and imaging

A more detailed model



Move to Chapter 3